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In-plane loading of a bonded rigid disc inclusion embedded at a pre-compressed elastic interface: The role of non-linear interface responses

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ABSTRACT

The paper examines the in-plane loading of a bonded rigid disc inclusion of finite thickness that is embedded at a smooth pre-compressed elastic interface composed of two halfspace regions. The in-plane loading induces a pure translation of the inclusion without rotation. The paper first develops estimates for the in-plane elastic stiffness of the disc inclusion. Upon de-bonding of the inclusion-elastic medium interfaces, the peak force is established by considering a Coulomb friction response of fully debonded interfaces. When the detached interface exhibits dilatant processes the displacement-dependent peak force at the detached surfaces can be estimated using a work-plastic energy dissipation relationship. When the bonded faces of the inclusion exhibit an elasto-plastic response, the force–displacement response of the in-plane loaded inclusion can exhibit a transition from the fully bonded response to partial detachment. The analysis of this problem is achieved using a computational approach that accounts for the development of failure and separation at the pre-compressed interface.

1. Introduction

Problems dealing with the mechanics of inclusions embedded in elastic media occupy an important position in materials engineering, composite materials, the modelling of interfaces of mechanical systems, geosciences and geomechanics. The inclusions are generally treated as regions of differing elastic properties and usually with complete contiguity between the inclusion and the surrounding elastic medium. The inclusions themselves are represented by either isolated ellipsoidal or spherical regions and the domain exterior to the inclusion is generally represented by a medium of infinite extent and material isotropy and elasticity features prominently in the treatment of inclusion problems. A complete exposition of the contributions to this field is beyond the scope of this article since this will entail a detailed review of several thousand articles. Some seminal contributions to the field of three-dimensional inclusion problems are given in [1–32]. Reviews of inclusion problems are also given in [33–38]. The class of problems that deal with imperfect contact between the three-dimensional inclusion and the surrounding elastic medium also has important applications in the study of effective properties of composites and in determining the conditions that promote failure and damage to multi-phasic elastic composites [38–44].

The disc inclusion is a special case of the three-dimensional inclusion with an ellipsoidal or spheroidal geometry and serves as a useful model for examining a simplified problem where the geometry of the inclusion allows its representation

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as a region with a planar form. The approaches to the study of disc inclusions rely largely on the mathematical formulation of allied problems in relation to the theory of integral equations that arise in the study of mixed boundary value problems in elasticity theory [45–55]. A large collection of problems involving disc inclusions with circular, annular, elliptical and other shapes have been examined in the literature. It is convenient to group these in relation to specific classes of problems, bearing in mind that some of the studies reported previously should be consulted to complete the list. The categories include (i) rigid or flexible, complete or annular disc inclusions in isotropic or transversely isotropic elastic, infinite or halfspace domains and subjected to direct forces or forces located in the medium exterior to the inclusion [56–81], (ii) disc inclusions located in extended domains or halfspace domains, with detached interfaces and interaction of disc inclusions with cracks [82–98], (iii) disc inclusions initiating unilateral contact [99–102], (iv) disc inclusions in bi-material and non-homogeneous regions [103–113] and (v) disc inclusions in poroelastic, piezo-ceramic and creep susceptible media [114–117]. It must be emphasized that the references cited are not meant to be a comprehensive review of the disc inclusion problem. The author has refrained from including articles with either incorrect or unrealistic assignment of boundary conditions relevant to the embedded disc inclusion problem.

This paper examines the in-plane force–displacement behaviour of a rigid disc inclusion of thickness $2h$ and radius a that is embedded in bonded contact at the smooth interface between two isotropic elastic halfspace regions, which are subjected to a pre-compression σ_0. The disc inclusion is subjected to an in-plane force P, which induces an in-plane translation Δ_h (Fig. 1).

The paper develops estimates for the in-plane elastic stiffness of the disc inclusion embedded at the interface. As the force P is increased there is detachment of the bonded interfaces of the inclusion and the detached surfaces can exhibit a variety of contact responses ranging from Coulomb friction to dilatant friction. In the case of an interface with Coulomb friction, the elastic halfspace regions containing the inclusion do not experience any additional displacements normal to the interface. When the faces of the disc inclusion exhibit dilatant friction, the relative movement at the inclusion-elastic halfspace contact also causes displacements normal to the plane of the rigid inclusion, which will induce additional asymmetric separation at the smooth pre-compressed interface, The extent of separation will depend on the pre-compression stress, the dilatant displacement and the elasticity characteristics of the halfspace regions. In this investigation, the interface separation is estimated by appeal to three-part boundary value problems applicable to axisymmetric problems related to an annular crack and the internal indentation of a penny-shaped crack. A relationship is developed for the post-interface failure load–displacement for the rigid disc inclusion that can also account for degradation of the dilatancy angle at the interfaces with progressive displacement of the inclusion. Finally, an incremental boundary element technique is used to develop the complete load–displacement relationship for the loaded inclusion when the interface exhibits elasto-plasticity phenomena as well as separation at the pre-compressed smooth interface.

2. The disc inclusion compressed between elastic halfspace regions

We consider the problem of a rigid disc of thickness $2h$ and radius a that is bonded to and compressed between two elastic halfspace regions with smooth plane boundaries and the halfspace regions by an axial stress σ_0 (Fig. 1). This will result in

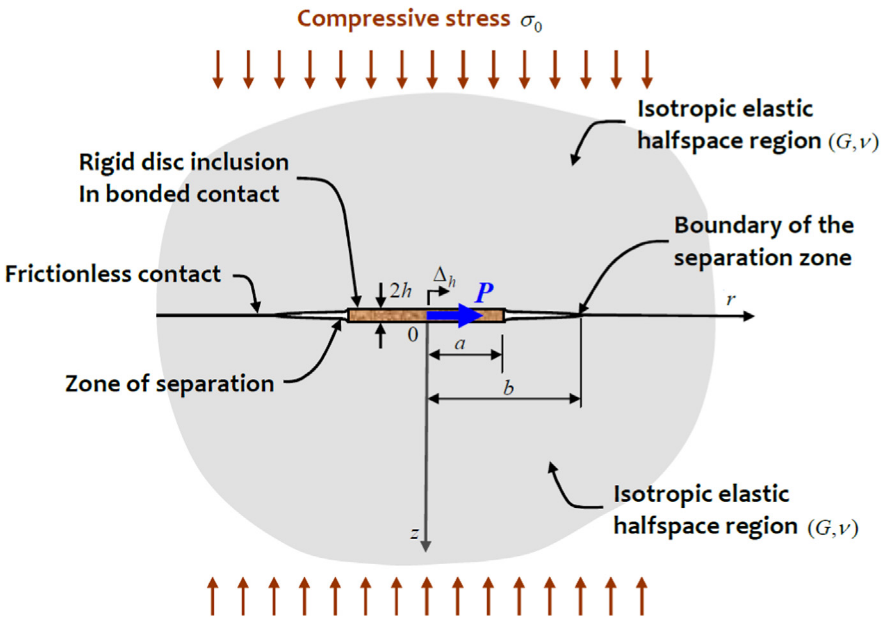

Fig. 1. Bonded rigid disc inclusion at a pre-compressed elastic interface.

smooth contact being established over the region $b < r < \infty$ where b is the radius of the zone of separation, which is unknown. Because of the symmetry of the problem, the mixed boundary value problem governing the contact problem can be defined in relation to a single halfspace region occupying $0 < r < \infty$ and $0 < z < \infty$, where (r, θ, z) is the cylindrical polar coordinate system. The axisymmetric mixed boundary value problem associated with the compressed inclusion problem can be formulated as follows:

$$\begin{aligned} u_r(r,0) &= 0 \quad ; \quad 0 \leqslant r \leqslant a \\ u_z(r,0) &= h \quad ; \quad 0 \leqslant r \leqslant a \\ u_z(r,0) &= 0 \quad ; \quad b \leqslant r \leqslant \infty \\ \sigma_{zz}(r,0) &= 0 \quad ; \quad a < r < b \\ \sigma_{rz}(r,0) &= 0 \quad ; \quad a < r < \infty \end{aligned} \tag{1}$$

where (u_r, u_θ, u_z) are the displacement components referred to the cylindrical polar coordinate system (r, θ, z) and $\boldsymbol{\sigma}(r, \theta, z)$ is the stress tensor referred to the cylindrical polar coordinate system given by

$$\boldsymbol{\sigma}(r,\theta,z) = \begin{bmatrix} \sigma_{rr} & \sigma_{r\theta} & \sigma_{rz} \\ \sigma_{r\theta} & \sigma_{\theta\theta} & \sigma_{\theta z} \\ \sigma_{rz} & \sigma_{\theta z} & \sigma_{zz} \end{bmatrix} \tag{2}$$

In addition to the mixed boundary conditions (1) the stress state should satisfy the far-field condition

$$\boldsymbol{\sigma}(r,z) \to \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\sigma_0 \end{bmatrix}_{z \to \infty} \tag{3}$$

At the boundary of the separation of contact between the halfspace regions, the contact normal stress should uniformly tend to zero and this constraint is used to estimate the boundary of separation b. The analysis of the mixed boundary value problem for determining the separation boundary can be approached by simplifying the first boundary condition of (1) and assuming that the disc inclusion is in frictionless contact with the elastic halfspace regions (Fig. 2).

The revised set of boundary conditions now reduce to

$$\begin{aligned} u_z(r,0) &= h \quad ; \quad 0 \leqslant r \leqslant a \\ u_z(r,0) &= 0 \quad ; \quad b \leqslant r \leqslant \infty \\ \sigma_{zz}(r,0) &= 0 \quad ; \quad a < r < b \\ \sigma_{rz}(r,0) &= 0 \quad ; \quad 0 < r < \infty \end{aligned} \tag{4}$$

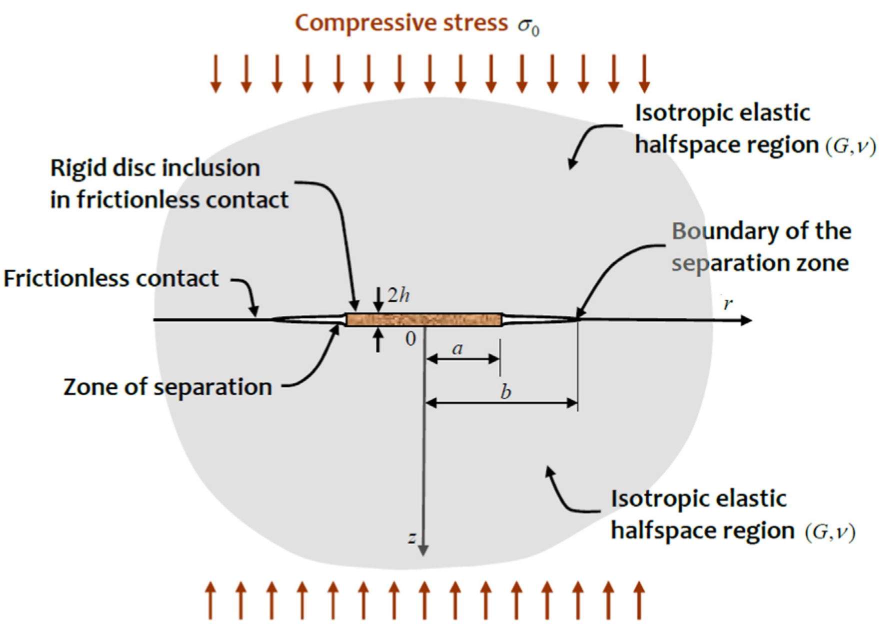

Fig. 2. Frictionless indentation of a disc inclusion by elastic halfspace regions.

The resulting unilateral problem can be examined as a combination of two three-part boundary value problems related to (i) the axisymmetric internal indentation of a penny-shaped crack of radius b by a smooth rigid disc inclusion of radius a thickness $2h$ and (ii) the internal pressurization of an annular crack of internal radius a and external radius b. Using Love's strain potential approach [52,118–121], this three part mixed boundary value problem lead to sets of triple integral equations [84,122]. The internal indentation problem gives rise to the system of equations

$$\begin{aligned} &\int_0^\infty \xi^{-1}R(\xi)J_0(\xi r)\,d\xi = -\tfrac{Gh}{(1-\nu)} \quad ; \quad 0 \leqslant r \leqslant a \\ &\int_0^\infty R(\xi)J_0(\xi r)\,d\xi = 0 \quad ; \quad a < r < b \\ &\int_0^\infty \xi^{-1}R(\xi)J_0(\xi r)\,d\xi = 0 \quad ; \quad b \leqslant r < \infty \end{aligned} \tag{5}$$

for the unknown function $R(\xi)$. The solution to this triple system can be obtained using a series approximation in terms of the parameter $c(= a/b) < 1$. Two results of interest to the inclusion problem are the resultant force generated due to the indentation of the penny shaped crack by the rigid disc inclusion and the Mode I stress intensity factor at the boundary of the penny-shaped crack. Omitting details, it can be shown that

$$P_h = 2\pi\int_0^a r\sigma_{zz}(r,0)dr = \frac{4aGh}{(1-\nu)}\left[\begin{array}{l} \left(1+\frac{4c}{\pi}\right)+\frac{16c^2}{\pi^4}+c^3\left(\frac{64}{\pi^6}+\frac{16}{9\pi^4}-\frac{8}{9\pi^2}\right) \\ +c^4\left(\frac{256}{\pi^8}+\frac{64}{9\pi^4}\right)+c^5\left(\frac{10240}{\pi^{10}}+\frac{9600}{225\pi^6}+\frac{92}{225\pi^2}\right)+\mathrm{O}(c^6) \end{array}\right] \tag{6}$$

where O( ) is the Landau symbol indicating the order of the approximation. The Mode I stress intensity factor at the boundary of the indented penny-shaped crack can be evaluated in the form

$$K_I^h = \frac{Gh}{\pi(1-\nu)\sqrt{b}}F_h(c) \tag{7}$$

where

$$F_h(c) = \left[\begin{array}{l} \frac{4c}{\pi}+\frac{16c^2}{\pi^3}+c^3\left(\frac{64}{\pi^5}+\frac{4}{3\pi}\right) \\ +c^4\left(\frac{80}{9\pi^3}+\frac{256}{\pi^7}\right)+c^5\left(\frac{448}{9\pi^5}+\frac{1024}{\pi^9}+\frac{4}{5\pi}\right)+\mathrm{O}(c^6) \end{array}\right] \tag{8}$$

Similarly, the problem of the internal pressurization of the annular crack gives rise to the system of triple integral equations

$$\begin{aligned} &\int_0^\infty S(\xi)J_0(\xi r)\,d\xi = 0 \quad ; \quad 0 \leqslant r \leqslant a \\ &\int_0^\infty \xi S(\xi)J_0(\xi r)\,d\xi = -\sigma_0; \quad a < r < b \\ &\int_0^\infty S(\xi)J_0(\xi r)\,d\xi = 0 \quad ; \quad b \leqslant r < \infty \end{aligned} \tag{9}$$

for the single unknown function $S(\xi)$. Here again, the solution can be developed in series form [123] and the force generated on the ligament region $0 < r < a$ can be expressed in the form

$$\begin{aligned} P_{\sigma_0} &= 2\pi\int_0^a \sigma_{zz}(r)rdr \\ &= \sigma_0\pi a^2\left(\begin{array}{l} -\frac{8}{\pi^2 c}+\left\{1-\frac{32}{\pi^4}\right\}+8c\left\{\frac{1}{\pi^2}-\frac{48}{\pi^6}\right\} \\ -\frac{c^2}{9\pi^8}\left\{4608+\pi^3(32-64\pi+3\pi^3)\right\} \\ -\frac{4c^3}{45\pi^{10}}\left\{23040+\pi^3(-320+480\pi+15\pi^3+6\pi^5)\right\} \\ -\frac{c^4}{675\pi^{12}}\left\{\begin{array}{l} 5529600+ \\ \pi^3\left(-76800+\pi\left[\begin{array}{l} 192000 \\ +\pi^2(3600+\pi\{-320+3168\pi+45\pi^3\}) \end{array}\right]\right) \end{array}\right\} \\ +\mathrm{O}(c^5) \end{array}\right) \end{aligned} \tag{10}$$

The Mode I stress intensity factor at the external boundary of the pressurized annular crack can be evaluated in the form

$$K_I^{\sigma_0} = \frac{2\sigma_0\sqrt{b}}{\pi}F_{\sigma_0}(c) \tag{11}$$

where

$$F_{\sigma_0}(c) = \left[\begin{array}{l} 1-\frac{4c}{\pi^2}-\frac{16c^2}{\pi^4}-c^3\left(\frac{1}{8}+\frac{64}{\pi^6}\right) \\ -c^4\left\{\frac{16}{3\pi^4}+\frac{4}{\pi^2}\left(\frac{1}{24}-\frac{8}{9\pi^2}+\frac{64}{\pi^6}+\frac{4}{9\pi^3}\right)\right\} \\ -c^5\left\{\frac{16}{\pi^4}\left(\frac{1}{24}-\frac{8}{9\pi^3}+\frac{64}{\pi^6}+\frac{8}{9\pi^2}\right)+\frac{256}{9\pi^6}-\frac{4}{15\pi^2}\right\}+\mathrm{O}(c^6) \end{array}\right] \tag{12}$$

The unknown radius of the separation zone can be estimated by assuming that the combined stress intensity factor given by (7) and (11) approaches zero at $r = b$: i.e.

$$\left(\frac{Gh}{\sigma_0 a(1-\nu)}\right)\frac{c}{2}F_h(c) - F_{\sigma_0}(c) = 0 \tag{13}$$

The lowest root of (13) gives the radius of the zone of separation b and this value can be used to estimate the resultant force generated on the disc inclusion due to the far-field compressive stress σ_0: i.e.

$$P_N = \sigma_0 \pi a^2 + \frac{4aGh}{(1-\nu)}P_N^h - \sigma_0 \pi a^2 P_N^{\sigma_0} \tag{14}$$

where

$$P_N^h = \begin{bmatrix} \left(1+\frac{4c}{\pi}\right) + \frac{16c^2}{\pi^4} + c^3\left(\frac{64}{\pi^6}+\frac{16}{9\pi^4}-\frac{8}{9\pi^2}\right) \\ +c^4\left(\frac{256}{\pi^8}+\frac{64}{9\pi^4}\right) + c^5\left(\frac{10240}{\pi^{10}}+\frac{9600}{225\pi^6}+\frac{92}{225\pi^2}\right) \end{bmatrix} \tag{15}$$

$$P_N^{\sigma_0} = \left(\begin{array}{l} -\frac{8}{\pi^2 c} + \left\{1-\frac{32}{\pi^4}\right\} + 8c\left\{\frac{1}{\pi^2}-\frac{48}{\pi^6}\right\} \\ -\frac{c^2}{9\pi^8}\left\{4608+\pi^3(32-64\pi+3\pi^3)\right\} \\ -\frac{4c^3}{45\pi^{10}}\left\{23040+\pi^3(-320+480\pi+15\pi^3+6\pi^5)\right\} \\ -\frac{c^4}{675\pi^{12}}\left\{5529600+\pi^3\left(-76800+\pi\begin{bmatrix}192000 \\ +\pi^2(3600+\pi\{-320+3168\pi+45\pi^3\})\end{bmatrix}\right)\right\} \end{array} \right) \tag{16}$$

The variation of b/a $(=1/c)$ with the non-dimensional parameter $Gh/\sigma_0 a(1-\nu)$is shown in Fig. 3.

In this condition, the plane interface between the rigid inclusion and the halfspace regions are in contact at the disc inclusion-elastic halfspace interface $0 \leqslant r \leqslant a$ and beyond the zone of separation $b \leqslant r < \infty$. The accuracy of the approximate analytical relationships for both the radius of separation and the resultant force generated on the disc inclusion has been verified by appeal to computational results [122].

Attention is now focused on the estimation of the elastic stiffness of the rigid disc inclusion that is embedded in bonded contact with the two halfspace regions (Fig. 1). This entails the solution of a mixed boundary value problem for the unilateral contact problem with the following boundary conditions:

$$\begin{array}{lll} u_r(r,\theta,0) = \Delta_h \cos\theta & ; & 0 \leqslant r \leqslant a \\ u_\theta(r,\theta,0) = -\Delta_h \sin\theta & ; & 0 \leqslant r \leqslant a \\ u_z(r,\theta,0) = h & ; & 0 \leqslant r \leqslant a \\ \sigma_{zz}(r,\theta) = 0 & ; & a < r < b(r,\theta) \\ \sigma_{rz}\sin\theta + \sigma_{\theta z}\cos\theta = 0 & ; & b(r,\theta) < r < \infty \\ \sigma_{rz}\cos\theta - \sigma_{\theta z}\sin\theta = 0 & ; & b(r,\theta) < r < \infty \end{array} \tag{17}$$

where $b(r,\theta)$is the unknown boundary of the separation zone. In addition to the mixed boundary conditions (17), the stress field should satisfy the far-field condition (3) and the boundary of the separation zone needs to be calculated through an iterative approach that involves a Signorini-type constraint [124,125], which entails an approximate approach involving

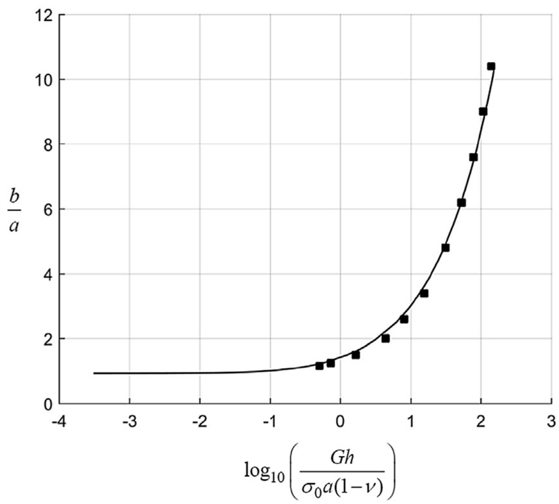

Fig. 3. Relationship between the boundary of the separated zone and the applied compressive stress σ_0.

two variables. The mixed boundary value problem described above is a non-trivial problem in elasticity theory and to date there is no solution available for this contact problem. The in-plane elastic stiffness of the disc inclusion can, however, be estimated as a *set of bounds* corresponding to reduced mixed boundary value problems. For the estimation of the in-plane stiffness we can idealize the problem where the disc inclusion is of zero thickness since the load transfer takes place primarily through the plane surfaces of the disc inclusion. As an *upper bound* for the stiffness we consider the case where the rigid disc inclusion of infinitesimal thickness is embedded in an elastic medium of infinite extent (Fig. 4) and the mixed boundary conditions

$$\begin{aligned} &u_r(r,\theta,0) = \Delta_h \cos\theta &&; \quad 0 \leqslant r \leqslant a \\ &u_\theta(r,\theta,0) = -\Delta_h \sin\theta &&; \quad 0 \leqslant r \leqslant a \\ &u_z(r,\theta,0) = 0 &&; \quad 0 \leqslant r \leqslant \infty \\ &\sigma_{rz}(r,\theta,0) = 0 &&; \quad a < r < \infty \\ &\sigma_{\theta z}(r,\theta,0) = 0 &&; \quad a < r < \infty \end{aligned} \tag{18}$$

The third boundary condition of (18) can be interpreted as the case where $(\sigma_0/G) \to \infty$, resulting in complete closure of the gap between the elastic halfspace regions shown in Fig. 1. The mixed boundary value problem described by (18) can be formulated using the generalized representations of the solution of elasticity problems in terms of harmonic and bi-harmonic functions [126] and the application of the procedures to both disc and annular inclusion problems is given in [73,75]. The harmonic and bi-harmonic functions are governed by

$$\nabla^2\Psi(r,\theta,z) = 0 \quad ; \quad \nabla^2\nabla^2\Phi(r,\theta,z) = 0 \tag{19}$$

where ∇^2 is Laplace's operator referred to the cylindrical polar coordinate system (r,θ,z) given by

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \tag{20}$$

Considering the displacement boundary conditions (18) the form of the functions $\Psi(r,\theta,z)$ and $\Omega(r,\theta,z)$ are

$$\Psi(r,\theta,z) = \left\{ \int_0^\infty \xi C(\xi) e^{-\xi z} J_1(\xi r) d\xi \right\} \sin\theta \tag{21}$$

and

$$\Omega(r,\theta,z) = \left\{ \int_0^\infty \xi [A(\xi) + zB(\xi)]\, J_1(\xi r) d\xi \right\} \cos\theta \tag{22}$$

where $A(\xi)$, $B(\xi)$ and $C(\xi)$ are arbitrary functions and $J_1(\xi r)$ is the first-order Bessel function. The mixed boundary conditions can be effectively reduced to a system of dual integral equations for a single unknown function $F(\xi)$

$$\begin{aligned} &\textstyle\int_0^\infty F(\xi) J_1(\xi r) d\xi = -\frac{4\Delta_h(1-\nu)}{(7-8\nu)} \quad ; \quad 0 \leqslant r \leqslant a \\ &\textstyle\int_0^\infty \xi F(\xi) J_1(\xi r) d\xi = 0 \quad ; \quad a < r < \infty \end{aligned} \tag{23}$$

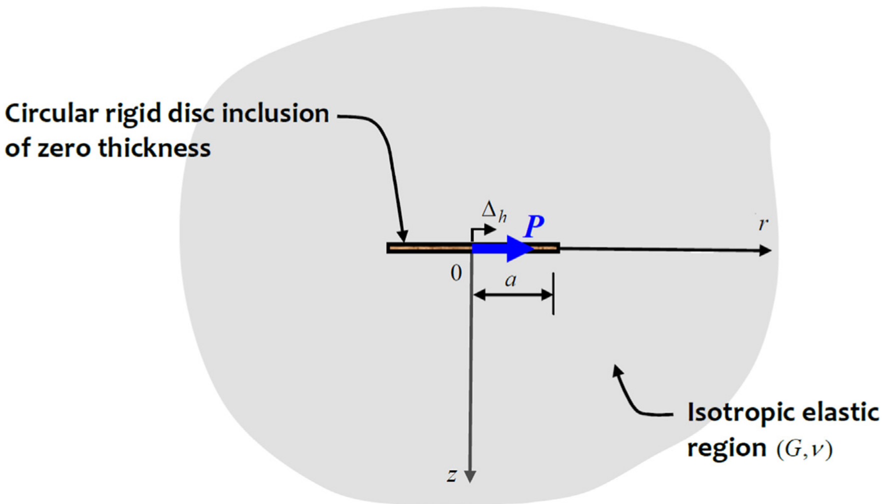

Fig. 4. A bonded rigid disc inclusion embedded in an elastic infinite space.

The solution of the dual system is given in several standard texts and articles on integral equations [52–55,120,121] and will not be pursued here. The result of interest to the study of the inclusion problem is the force–displacement relationship, which can be evaluated in exact closed form: i.e.

$$P = \frac{64(1-\nu)Ga\Delta_h}{(7-8\nu)} \tag{24}$$

where G is the linear elastic shear modulus and ν is Poisson's ratio of the elastic medium. The lower bound estimate for the in plane translational stiffness of the disc inclusion can be obtained by assuming that $(\sigma_0/G) \to 0$, in which case the disc inclusion is embedded in bonded contact with two non-interacting halfspace regions (Fig. 5).

Considering a single halfspace region ($z \geqslant 0$), the mixed boundary conditions associated with the contact problem is given by

$$\begin{aligned} u_r(r,\theta,0) &= \Delta_h \cos\theta \quad ; \quad 0 \leqslant r \leqslant a \\ u_\theta(r,\theta,0) &= -\Delta_h \sin\theta \quad ; \quad 0 \leqslant r \leqslant a \\ \sigma_{zz}(r,\theta,0) &= 0 \quad ; \quad 0 \leqslant r \leqslant \infty \\ \sigma_{rz}(r,\theta,0) &= 0 \quad ; \quad a < r < \infty \\ \sigma_{\theta z}(r,\theta,0) &= 0 \quad ; \quad a < r < \infty \end{aligned} \tag{25}$$

The mixed boundary value problem defined by (25) has been examined in connection with adhesive contact problems for a halfspace region and the complete formulation requires the solution of a Hilbert problem and gives rise to oscillatory singularities at the boundary of the disc inclusion that are integrable. Details of the methods of solution are given in [52,54,109,127,128]. The exact closed form lower bound for the in-plane load–displacement relationship for the disc inclusion takes the form

$$P = \frac{16Ga\Delta_h}{\left\{1 + \frac{(1-2\nu)}{\ln(3-4\nu)}\right\}} \tag{26}$$

In the limit of incompressibility ($\nu \to 1/2$), both (24) and (26) reduce to the identical result $P = 32Ga\Delta_h/3$. In the limit when $\nu \to 0$

$$\frac{[P/32Ga\Delta_h]_{\text{Eq}(24)}}{[P/32Ga\Delta_h]_{\text{Eq}(26)}} \simeq 1.09 \tag{27}$$

Alternative approaches involve solving the mixed boundary value problem where the oscillatory form of the singularity is replaced by a regular $1/\sqrt{r}$ type singularity. It has been shown [75–77,87–91,98,110,111] that in terms of the evaluation of the load–displacement relationships, this approach gives numerical results that are accurate to within 0.04% of the exact analytical solution when $\nu = 0$ [77]. A further result of interest to the estimation of the in-plane force–displacement relationship involves the loading of a disc inclusion embedded in bonded contact within a penny-shaped crack (Fig. 6) [96].

The results show that when the location of the external boundary $b > 10a$, and $\nu = 0$, the influence of the contact region beyond the external boundary has virtually no influence on the in-plane load–displacement of the disc inclusion. The elastic

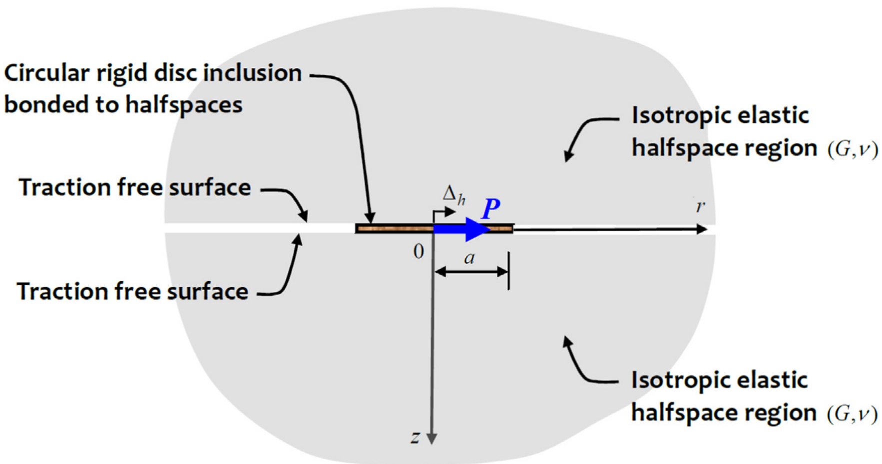

Fig. 5. A bonded rigid disc inclusion embedded between non-interacting elastic half spaces.

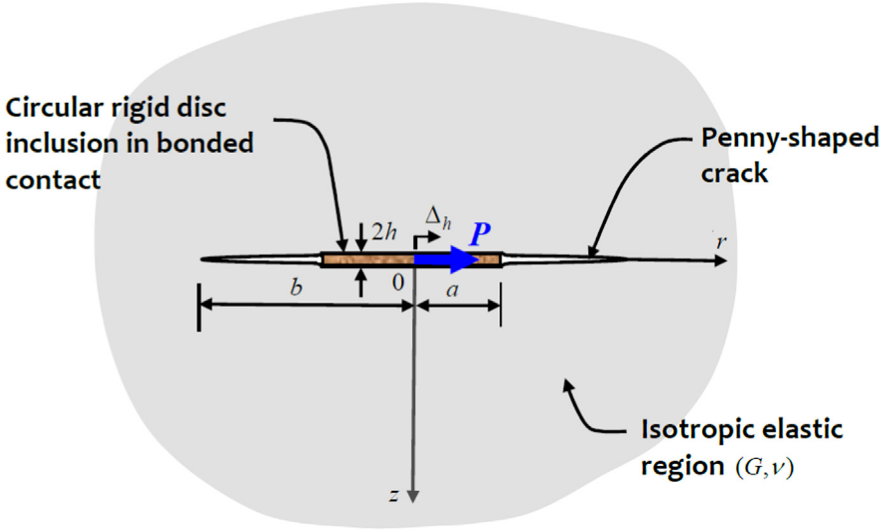

Fig. 6. In-plane loading of a rigid disc inclusion embedded in a penny-shaped crack.

stiffness of the disc inclusion can be estimated using the analytical estimates given by either (24), (26) or the numerical results given in [96].

The load–displacement relationships (24) and (26) are valid provided the interface between the elastic media and the disc inclusion does not experience any failure. In the case when the disc inclusion-elastic halfspace interface exhibits post-failure Coulomb frictional phenomena, the limiting value of the in-plane force P_C is directly related to the peak frictional force that can be generated at the interface, which in turn will depend on (i) the far-field normal stress σ_0 acting on the interface containing the rigid circular disc inclusion, (ii) the normal stress induced in the bonded inclusion region due to the lateral translation and (iii) the coefficient of friction at the interface. Since the axial stress induced by the lateral translation of the embedded inclusion is asymmetric in θ the resultant induced normal force is zero. The *peak resultant in-plane force* acting on the plane faces of the disc inclusion due to the initial stress σ_0 is

$$P_C = 2\left(\sigma_0 \pi a^2 + \frac{4aGh}{(1-\nu)} P_N^h - \sigma_0 \pi a^2 P_N^{\sigma_0}\right) \tan\varphi \tag{28}$$

where φ is the angle of friction. In the case where the compressive stress σ_0 is applied to a disc inclusion of zero thickness located at the interface, (28) for the peak in-plane shear load gives a result where the normal stress is reduced by the contribution of $P_N^{\sigma_0}$. If full contact between the halfspace regions occurs exterior to the disc inclusion (i.e. $c \to 1$), the series approximation will give rise to $P_N^{\sigma_0} \simeq 0.0272$. When the far-field compressive stress $\sigma_0 \to 0$, the normal force obtained from (28) will correspond to that generated by the frictionless Boussinesq indentation of the disc inclusion on the halfspace regions [47–55,119–121,125–130].

2.1. The inclusion-elastic media interaction: the dilatant interface

The results for the indentation of an interface by the rigid inclusion and its influence on the peak in-plane load that can be applied to it will be influenced by the contact properties of the interface. The result (28) can also be extended to include dilatancy effects that can result from the contact characteristics of the inclusion-elastic medium interface, which leads to displacements normal to the direction of relative in-plane movements. The topic of contact mechanics of interfaces in itself has a rich history dating back to Leonardo da Vinci, Amontons, Desaguliers and others and historical perspectives of the subject with applications are given in [52–55,122,131–139] When post failure dilatancy effects are present at the interface, the in-plane movement of the inclusion Δ_h (Fig. 7) induces a displacement Δ_n normal to the plane of movement.

The analysis of the shear failure problem during the generation of frictional phenomena and dilatancy effects should be examined by appeal to a theory of plasticity applicable for elasto-plastic phenomena with specified failure criteria and non-associated flow rules [140–143]. In the presence of interface dilatancy, the peak load that can be applied to embedded inclusions can be determined using the work-energy dissipation relationship procedure developed for the analysis of dilatancy processes in granular materials [144,145], where the stored elastic energy at the interface is neglected. This argument is consistent with the limit analysis concepts proposed in the literature [122,141,145]. The basic approach involves the relationship obtained by equating the work done by the shearing forces and normal forces to the energy dissipated at the frictional-dilatant region, expressed in terms of force resultants rather than exact distributions over the contact zone. The work com-

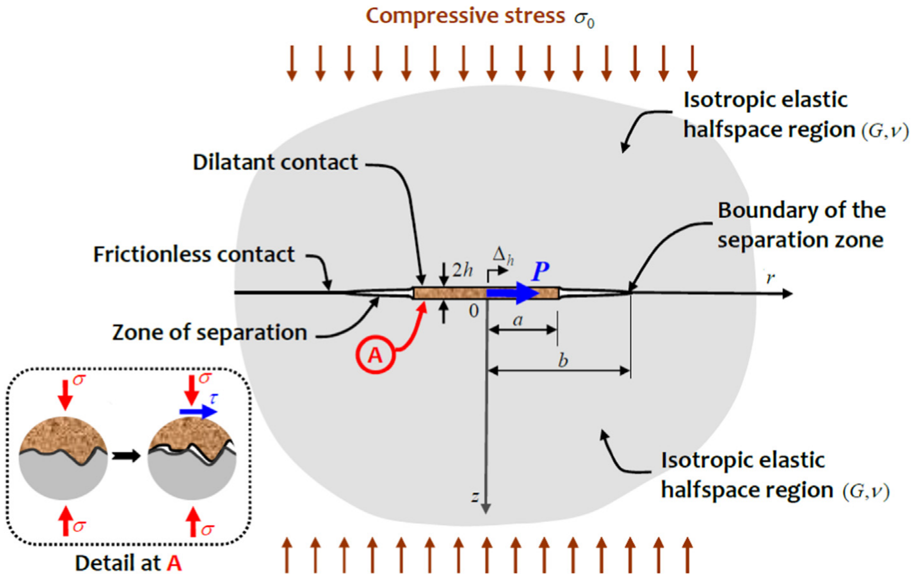

Fig. 7. Interface dilatancy between the rigid disc inclusion and the elastic half spaces.

ponent Wconsists of the work of the peak shear force (P_D) acting at the onset of rupture and the work of the normal force (P_N^D) induced by both the initial indentation (h) of the inclusion and dilatancy on the upper and lower surfaces: i.e.

$$W = P_D(\Delta_h) + 2P_N^D(-\Delta_n) \tag{29}$$

where

$$P_N^D = \left\{\sigma_0 \pi a^2 + \frac{4aGh}{(1-\nu)} P_N^h \left(1 + \frac{\Delta_h}{h} \tan\alpha\right)\right\} \tag{30}$$

and α is the dilatancy angle and $\tan\alpha = \Delta_n/\Delta_h$. Also, in (30), the error term $\sigma_0 \pi a^2 P_N^{\sigma_0}$ is neglected. The energy dissipated at the dilatant circular patch is given by

$$D = 2P_N^D\,(\Delta_h)\tan\varphi \tag{31}$$

From (29) and (31) we obtain the following result for the peak load that is generated when the disc inclusion-elastic media interfaces exhibit dilatancy characteristics:

$$P_D = 2\left\{\sigma_0 \pi a^2 + \frac{4aGh}{(1-\nu)} P_N^h \left(1 + \frac{\Delta_h \tan\alpha}{h}\right)\right\}(\tan\alpha + \tan\varphi) \tag{32}$$

In the absence of dilatancy phenomena, $\alpha \to 0$ and (32) reduces to the result where the interface response is influenced only by Coulomb friction. It is also important to note that in the case of interface dilatancy, the in-plane force generated will be influenced by the magnitude of the in-plane displacement Δ_h. As noted in many studies involving tribology, materials science and the geosciences, the dilatancy angle α can degrade with the parameter Δ_h/h. The method presented here can be extended to cover the degradation of the dilatancy angle α [122]. Since the expression for the interface load exhibiting frictional-dilatancy phenomena is evaluated in exact closed form, the result can be conveniently evaluated for any specified set of parameters relevant to the disc inclusion problem.

It must, however, be noted that the development of the peak forces P_C and P_D has an incremental force–displacement history, in the sense that the detachment of the bonding and the development of friction in the detached regions occurs with the progressive increase in the in-plane force. This aspect is best examined using a computational approach. In such a computational approach, the boundary of the bonded-debonded regions needs to be determined by appeal to a failure criterion applicable to the inclusion-elastic medium interface. An approach for the modelling of the inclusion with an elasto-plastic interface is discussed in a subsequent section.

3. The inclusion-elastic media interaction: the elasto-plastic interface

We consider the problem where the interface between the elastic medium and the disc inclusion exhibits elasto-plastic phenomena. When the elasto-plastic phenomena are restricted only to the interface, it is convenient to adopt an incremental boundary element formulation of the problem. Approaches to the conventional incremental boundary element formulation of contact problems are documented in several contributions [145–150]. Hybrid boundary element approaches [151,152] can also be extended to examine effects of interface non-linearity.

For isotropic elastic media the incremental form of the boundary integral equation governing an individual region $D^{(\rho)}$ can be written as

$$c_{ij}\dot{u}_j^{(\rho)} + \int_{\Gamma^{(\rho)}} P_{ij}^{*(\rho)}\dot{u}_j^{(\rho)} d\Gamma = \int_{\Gamma^{(\rho)}} u_{ij}^{*(\rho)}\dot{P}_j^{(\rho)} d\Gamma \tag{33}$$

where $i,j = 1,2,3$ or (x,y,z); the Greek superscript or subscript refers to the material region; $\dot{u}_j^{(\rho)}$ and $\dot{P}_j^{(\rho)}$ are the incremental values of the boundary displacements and tractions respectively; $u_{ij}^{*(\rho)}$ and $P_{ij}^{*(\rho)}$are the corresponding fundamental solutions given by

$$u_{ij}^{*(\rho)} = \frac{1}{16\pi G_\rho(1-\nu_\rho)r}\left[(3-4\nu_\rho)\delta_{ij} + r_{,i}r_{,j}\right] \tag{34}$$

and

$$P_{ij}^{*(\rho)} = -\frac{1}{8\pi(1-\nu_\rho)r^2}\left\{\left[(1-2\nu_\rho)\delta_{ij} + 3r_{,i}r_{,j}\right]r_{,n} - (1-2\nu_\rho)\left[r_{,i}n_j - r_{,j}n_i\right]\right\} \tag{35}$$

where r is the distance between the source and the field points; n_i are the components of the outward unit normal vector to $\Gamma^{(\rho)}$; δ_{ij} is Kronecker's delta function; G_ρ and ν_ρ are , respectively, the linear elastic shear modulus and Poisson's ratio for the region ρ. Also, in (33), $c_{ij} = \delta_{ij}/2$ if the boundary is smooth. Results (33) to (35) can be developed for each sub-region $D^{(\rho)}$ $(\rho = 1,2,....,m)$ of a multi-domain separated by interfaces that exhibit non-linear material properties on interfaces $S_{(\beta)}$ $(\beta = 1,2,....,n)$. The regions in contact can be subjected to the following types of interface conditions:

(a) Prescribed displacements on boundary S_1 where

$$\dot{u}_i = \dot{u}_i^0 \tag{36}$$

(b) Prescribed tractions on boundary S_2 where

$$\dot{P}_i = \dot{P}_i^0 \tag{37}$$

(c) Interface conditions on boundary $S_2$where

$$\dot{P}_i = \dot{R}_i + K_{ij}^*\dot{u}_j \tag{38}$$

where $\dot{R}_i$ are incremental residual tractions and K_{ij}^* are stiffness coefficients considering non-linear processes at interfaces defined by frictional, elastic–plastic, dilatant processes. Following conventional procedures involving boundary element discretizations, the Eq. (33) can be reduced to a matrix equation of the form

$$[\mathbf{H}]\{\dot{\mathbf{u}}\} = [\mathbf{G}]\left\{\dot{\mathbf{P}}\right\} \tag{39}$$

where $[\mathbf{H}]$ and $[\mathbf{G}]$ are boundary element influence coefficients matrices. If the configuration of the boundary and the interface conditions can be defined at any level of deformations, the final incremental form of the matrix equation governing the boundary element formulation can be expressed in the generalized form

$$[\mathbf{A}]\left\{\dot{\mathbf{U}}\right\} = \left\{\dot{\mathbf{B}}\right\} \tag{40}$$

The mechanical behavior of an interface can be controlled by a variety of micro- and macro-structural phenomena that can result in failure, fracture, de-lamination, damage, elastic–plastic phenomena, dilatancy and degradation. The studies in this are quite extensive and no attempt will be made to provide a comprehensive review. Advances in this area are summarized in [52,122,131,132–139]. In this study, we focus attention on interfaces that can exhibit Coulomb friction or dilatant friction with interface degradation that can result from asperity breakage. The interface constitutive relationships are defined in terms of the incremental relative displacement at a contact zone $\dot{\Delta}_i$ and the corresponding incremental tractions $\dot{t}_i$. Following the conventional procedures for the formulation of problems in incremental plasticity [140–142], we assume that the incremental relative displacement $\dot{\Delta}_i$ is composed of an elastic recoverable component $\dot{\Delta}_i^{(e)}$and an irrecoverable or plastic component $\dot{\Delta}_i^{(p)}$: i.e.

$$\dot{\Delta}_i = \dot{\Delta}_i^{(e)} + \dot{\Delta}_i^{(p)} \tag{41}$$

We note that for an interface either i (or j) can be assigned notations applicable to the local interface coordinates. For ease of reference, we shall denote $i,j = x,y,z$ and the normal to the interface is assigned the z-coordinate. The elastic component of (41) is related to the interface tractions through the linear constitutive law

$$\dot{t}_i = k_{ij}\dot{\Delta}_j^{(e)} \tag{42}$$

where k_{ij} are the linear elastic stiffness coefficients of the interface. The irreversible components of (41) can be obtained by specifying the type of non-linear constitutive model applicable to the interface.

3.1. The interface exhibiting Coulomb friction

The yield function F for an interface exhibiting Coulomb friction can be written as

$$F = \left(t_x^2 + t_y^2\right)^{1/2} + t_z \tan\varphi = 0 \tag{43}$$

where t_i are the total values of the tractions and φ is the angle of friction. When the failure condition (43) is attained the interface will experience relative slip and the irreversible slip displacement can be obtained from a flow/slip rule similar to that used in the classical theory of plasticity [140–142] i.e.

$$\dot{\Delta}_i^{(p)} = \dot{\lambda}\frac{\partial\Phi}{\partial t_i} \tag{44}$$

where $\dot{\lambda}$ is a proportionality factor or the plastic flow/slip multiplier and Φ is the plastic flow/slip potential given by

$$\Phi = \left(t_x^2 + t_y^2\right)^{1/2} \tag{45}$$

Substituting (44) into (41) and then into (42) we have

$$\dot{t}_i = k_{ij}\left(\dot{\Delta}_j - \dot{\lambda}\frac{\partial\Phi}{\partial t_i}\right) \tag{46}$$

When the interface displacements experience slip, the incremental yield function gives the consistency condition

$$\frac{\partial F}{\partial t_i}\dot{t}_i = 0 \tag{47}$$

Using (46) and (47) we obtain

$$\dot{\lambda} = \frac{1}{\psi}\frac{\partial F}{\partial t_i}k_{ij}\dot{\Delta}_j ; \psi = \frac{\partial F}{\partial t_l}k_{lm}\frac{\partial\Phi}{\partial t_m} \tag{48}$$

Assuming that the irreversible part of the incremental deformations $\dot{\Delta}_i^{(p)}$ is governed by a relationship of the type

$$\dot{t}_i = k_{ij}^{(ep)}\dot{\Delta}_j \tag{49}$$

it can be shown that

$$k_{ij}^{(ep)} = k_{ij} - \frac{1}{\psi}\frac{\partial\Phi}{\partial t_l}k_{il}k_{mj}\frac{\partial F}{\partial t_m} \tag{50}$$

Following conventional procedures for the implementation of classical associative or non-associative plasticity phenomena, once the yield function (F) and the slip potential (Φ) are known, it is possible to define $k_{ij}^{(ep)}$. In the case when F and Φ are given by (43) and (45) respectively, and for the specific instance when

$$k_{xx} = k_{yy} = k_t \quad ; \quad k_{zz} = k_n \tag{51}$$

and all other $k_{ij} = 0$, (50) can be expressed in the form

$$[\mathbf{k}]^{(ep)} = \frac{1}{\left(t_x^2 + t_y^2\right)}\begin{pmatrix} k_t t_y^2 & -k_t t_x t_y & -k_n t_x \left(t_x^2 + t_y^2\right)^{1/2}\tan\varphi \\ -k_t t_x t_y & k_t t_x^2 & -k_n t_y \left(t_x^2 + t_y^2\right)^{1/2}\tan\varphi \\ 0 & 0 & k_n\left(t_x^2 + t_y^2\right) \end{pmatrix} \tag{52}$$

The elasto-plastic modelling of the interface can be extended to other types of responses. For example when the interface exhibits dilatant phenomena characterized by a friction angle φ and a dilatancy angle α the relevant forms for the yield function F and the plastic potential Φ are as follows:

$$F = \left[\left\{ t_x \sin\alpha + \left(t_x^2 + t_y^2 \right)^{1/2} \cos\alpha \right\}^2 \right]^{1/2} + \left[t_x \cos\alpha - \left(t_x^2 + t_y^2 \right)^{1/2} \sin\alpha \right] \tan\varphi$$
$$\Phi = \left[t_x \sin\alpha + \left(t_x^2 + t_y^2 \right)^{1/2} \cos\alpha \right]^{1/2} \tag{53}$$

These reduce to the Coulomb friction model when the dilatancy angle $\alpha = 0$.

3.2. Contact and separation processes

The basic approach outlined in the previous section can be implemented in an incremental boundary element scheme giving due consideration to the development of separation, re-contact, slip and adhesion

(a) *Separation*: During a loading process it is likely that the normal contact traction at the interface becomes tensile. In such a case the boundaries at the interface will separate and the boundary condition will be of the S_2-type (Eq. (37)).

(b) *Re-contact*: Also, re-contact is possible when the relative displacement across the separated interface is greater than the initial gap (the Signorini-constraint) In this case the boundary condition in the direction of re-contact will be changed from an S_2-type to an S_1-type (Eq. (36)).

(c) *Slip*: The condition for slip is given by the yield function (43) and when this condition is reached, the expression in (52) will be applied as an S_2-type boundary condition (Eq. (38)).

(d) *Adhesion*: When the stress condition at the interface does not violate the yield criterion (43), the boundary condition at the interface will be of the S_3-type (Eq. (42)).

With each increment and iteration, all of the above four conditions are checked until a stable convergent result is obtained.

3.3. Localized iterative solution procedure

For the boundary conditions (36) to (38), we can re-write the matrix Eq. (39) in the form

$$\left[-\mathbf{G}^{(1)},\ -\mathbf{H}^{(2)}, \left\{ -\mathbf{H}^{(3)} - \mathbf{G}^{(3)}\mathbf{K}^{(ep)} \right\} \right] \begin{Bmatrix} \dot{t}^{(1)} \\ \dot{\Delta}^{(2)} \\ \dot{\Delta}^{(3)} \end{Bmatrix} = \left[-\mathbf{H}^{(1)}, \mathbf{G}^{(2)}, \mathbf{G}^{(3)} \right] \begin{Bmatrix} \dot{\Delta}^{0(1)} \\ \dot{t}^{0(2)} \\ \dot{R}^{(3)} \end{Bmatrix} \tag{54}$$

where the superscripts $^{(i)}$, $i = 1, 2, 3$ indicate the relevant boundary condition. For the non-linear interface problem, we need to apply an efficient solution scheme to analyze the incremental and iterative matrix Eq. (54). Several such techniques are reported in the literature [141–144] and the scheme adopted here can be summarized in the following.

Since the boundary consists of linear and non-linear constraints, we can apply the elimination to the linear portion of the boundary constraint

$$\left[\tilde{\mathbf{A}},\ \tilde{\mathbf{H}}^{(3)} - \mathbf{G}^{(3)}\mathbf{K}^{(ep)} \right] \begin{Bmatrix} \dot{t}^{(1)} \\ \dot{\Delta}^{(2)} \\ \dot{\Delta}^{(3)} \end{Bmatrix} = \left\{ \tilde{\mathbf{B}} \right\} + \left[\tilde{\mathbf{G}}^{(3)} \right] \left\{ \dot{R}^{(3)} \right\} \tag{55}$$

where $\left[\tilde{\mathbf{A}} \right]$ is the reduced version of $\left[-\mathbf{G}^{(1)},\ \mathbf{H}^{(2)} \right]$ and it is an upper triangle type matrix; $\left\{ \tilde{\mathbf{B}} \right\}$is the reduced form of the right hand side vector from known boundary values and $\left[\tilde{\mathbf{H}}^{(3)} \right]$ and $\left[\tilde{\mathbf{G}}^{(3)} \right]$are the corresponding reduced forms. The result (55) can be separated into two relations: the first relationship corresponding to (S_1) and (S_2) is

$$\begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{pmatrix} \begin{pmatrix} \dot{t}^{(1)} \\ \dot{\Delta}^{(2)} \end{pmatrix} = \begin{pmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{pmatrix} + \begin{pmatrix} \tilde{G}_1^{(3)} \\ \tilde{G}_2^{(3)} \end{pmatrix} \left\{ \dot{R}^{(3)} \right\} - \begin{pmatrix} \tilde{H}_1^{(3)} & -\tilde{G}_1^{(3)} K^{(ep)} \\ \tilde{H}_2^{(3)} & -\tilde{G}_2^{(3)} K^{(ep)} \end{pmatrix} \left\{ \dot{\Delta}^{(3)} \right\} \tag{56}$$

which is a back-substitution form of the solution of $\{\dot{t}^{(1)}\}$ and $\left\{ \dot{\Delta}^{(2)} \right\}$ if $\left\{ \dot{\Delta}^{(3)} \right\}$is known. It may noted that that (56) can be used at any level of an increment when the boundary condition on $(S_2)$is determined. The second relationship is an uncoupled equation for $\left\{ \dot{\Delta}^{(3)} \right\}$; i.e.

$$\begin{pmatrix} \tilde{\mathbf{H}}_3^{(3)} & -\tilde{\mathbf{G}}_3^{(3)} \mathbf{K}^{(ep)} \end{pmatrix} \left\{ \dot{\Delta}^{(3)} \right\} = \left\{ \tilde{\mathbf{B}} \right\} + \left[\tilde{\mathbf{G}}_3^{(3)} \right] \left\{ \dot{R}^{(3)} \right\} \tag{57}$$

which has the unknowns only on the boundary (S_3). Thus, at any increment level (57) can be applied in an iterative manner in order to determine a configuration of the boundary (S_3). The non-linear problem is solved by a localized iterative procedure and the overall boundary element method system is eliminated only once for any number of increments.

4. Comparison of analytical and computational results for the in-plane loading of the disc inclusion

We now consider the computational modelling of the problem of a rigid disc inclusion of thickness $2h$ and radius a that is compressed between the two elastic halfspace regions by an axial stress σ_0 and subjected to an in-plane force P (Fig. 1). The problem is examined in two stages. (i) When examining the initial effects due to the application of the compressive stress state σ_0, both the contact between the disc inclusion and the elastic half spaces and the regions exterior to the disc inclusion are assumed to be frictionless. The effects of bonding at the interfaces and frictional effects in the exterior contact regions can be incorporated but this entails the analysis of a completely separate contact problem. The boundary element scheme is used only to identify the region of separation exterior to the disc inclusion resulting from the compression. (ii) Coulomb frictional phenomena at the contact between the inclusion and the elastic halfspace regions are allowed to materialize when the disc inclusion is subjected to the in-plane force P. During the application of the in-plane force the boundary of the frictionless contact region between the halfspace regions can change with the development of a boundary of separation $b(r,\theta)$. In the computational treatment of the frictional phenomena, this boundary is kept to the value derived from the initial compression. To perform the non-linear boundary element analysis of the in-plane loading of the disc inclusion it is necessary to specify the interface stiffness elements defined by (51) and other parameters relevant to defining the Coulomb friction model. The normalized values for the interface stiffnesses k_n , k_t and the Coulomb friction μ are specified as

$$\frac{k_t a}{G} = 10 \quad ; \quad \frac{k_n a}{G} = 10^7 \quad ; \quad \mu = \tan 30^0 \tag{58}$$

The linear elastic shear modulus is used as a normalizing parameter and Poisson's ratio of the elastic halfspace regions are assigned $v = 0.30$. The geometry of the disc inclusion is specified as $(2h/a) = 0.02$. The boundary element procedure is used to determine the normalized in-plane force $(P/\pi G a^2)$ vs. the normalized in-plane displacement (Δ_h/a) relationship. Fig. 8 shows the results for the normalized in-plane force vs, the normalized in-plane displacement obtained from the incremental boundary element approach. The elasticity solutions for the in-plane stiffness obtained from analytical approaches summarized by (24) and (26) are also shown for purposes of comparison. In general, there is good agreement between the computational and analytical results. Fig. 8 also shows the normalized peak Coulomb force that can be applied to the embedded, pre-compressed load as determined from (28). The results are in good agreement with the peak load estimated using the boundary element technique, to within an accuracy of less than 1%. As the pre-compression increases there is an attendant increase in the peak Coulomb load capacity of the embedded disc inclusion. The interface stiffness coefficients given by (58) are not expected to exert an influence on the elastic stiffness and the peak Coulomb load that can be sustained by the disc inclusion embedded at the pre-compressed elastic interface. The role of interface plasticity materializes over a very limited range of the normalized load displacement response of the disc inclusion. For all intents and purposes, the load–displacement behaviour of the embedded disc inclusion can be approximated by an elastic-rigid plastic model where the elastic stiffnesses and the peak loads can be estimated using the results presented in the paper.

5. Prospects for experimental simulations

The contact problem examined in the paper addresses several issues ranging from frictional and dilatant phenomena at interfaces, the development of receding/advancing unilateral contacts resulting from the compression of the disc inclusion between halfspace regions and bonded contact conditions between the halfspace regions and the disc inclusions. Experimental simulations of the problem can be developed to examine the various types of interaction problems involving half space regions fabricated with elastic halfspace domains that will maintain their linear elastic character without the development of contact fracture during compression of the rigid inclusion [153]. Several geological materials or high strength alloys can be used for the simulation of the halfspace regions. The manner of application of the load to the embedded disc inclusion can vary with the objective of the test. If a rigid rod is used to apply the load, the frictionless contact between the loading rod and the compressed halfspace regions need to be assured through the provision of Teflon coating of the rod. A twin-rod loading arrangement can be used to conduct load reversal tests where interface degradation can also be observed, leading to the reduction in either the coefficient or the dilatancy angle.

Of related interest is the possible application of the embedded disc inclusion problem that can serve as mechanical system that can be used to provide elastic restraint derived from the deformation of rubber-like elastic interfaces. The in-plane movement of the embedded disc inclusion will induce large deformations in the rubber-like elastic solids [154–161] and the spatial symmetry associated with the deformation will be such that the linearity in the load vs. in-plane displacement of the disc inclusion will be valid for moderately large elastic deformations that are characterized by the theory of second-order elasticity [162–172].

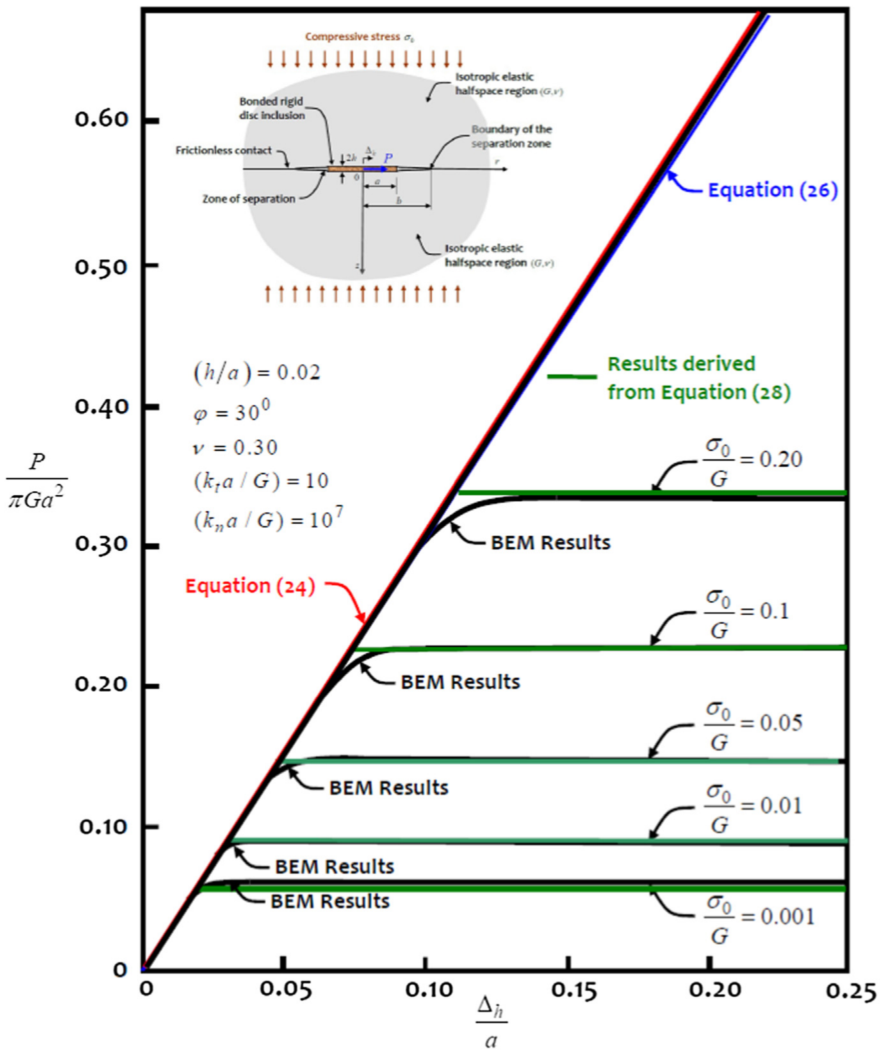

Fig. 8. Results for the normalized load vs. the normalized in-plane displacement.

The topic of frictional contact between structural components and their influence on the dynamics of structures [173] has been extensively investigated by Professor L. Gaul and co-workers [135, 174–180]. These experimental simulations involving compression of frictional inclusions are relegated to further studies.

6. Concluding remarks

Solutions derived from inclusion problems are used quite extensively for the modelling of composites and multiphasic materials, earth sciences and in geomechanics. The interfaces between inclusions and the surrounding elastic media are generally treated as being contiguous and extensions can be made to include effects of delamination, debonding and frictionless interfaces that can experience unilateral contact. These processes are generally non-linear and the analysis of the resulting inclusion problem can be attempted only through the use computational approaches. The paper develops solutions to the elastic stiffness and peak Coulomb load development of a rigid circular disc inclusion that is located at an otherwise smooth pre-compressed elastic interface. Analytical results developed are compared with results derived from and incremental boundary element analysis of the same problem. It is shown that in terms of engineering applications of the movement of the embedded disc inclusion, an idealized elastic perfectly plastic result can be used to describe the in-plane load–displacement response. The effects of interface plasticity have the effect of generation a non-linear transition between the elastic and perfectly plastic results based on the Coulomb friction model. If the elastic perfectly plastic model is adopted, the results lend themselves to the consideration of cases involving load reversal and the estimation of plastic energy dissipation via a simplified analysis.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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