On the asymmetric indentation of a consolidating poroelastic half space

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This paper examines the asymmetric indentation of a poroelastic half space by a rigid smooth circular indentor that is subjected to a moment resultant about a diametral axis. Three types of drainage conditions at the surface of the poroelastic half space are considered. The paper develops the governing coupled integral equations and reduces them to systems of Fredholm integral equations of the second kind in the Laplace transform domain. Efficient computational algorithms are proposed to evaluate the time-dependent behavior of the rigid circular foundation. The influence of the drainage boundary conditions and the compressibility of the pore fluid on the consolidation-induced rotation of the rigid circular foundation is discussed.

Keywords: consolidation, settlement, indentation, poroelasticity, integral equations, half space

1. Introduction

A one-dimensional model for the consolidation of a fluid-saturated medium was first proposed by Terzaghi. The model proposed by Terzaghi was extended by Biot to develop the now classical model of poroelasticity for a fluid-saturated medium. This model assumes isotropic linear elastic behavior of the soil skeleton, Darcy flow behavior of the pore fluid, and compressibility of the pore fluid. Biot's model of poroelasticity has been successfully applied to the study of three-dimensional soil consolidation problems in geomechanics. Biot and Clingan applied the model of poroelasticity to the study of the interaction between a slab and a consolidating foundation. McNamee and Gibson considered both axisymmetric and plane strain problems related to the consolidation of a half space region by a uniform surface load. Schiffman and Fungaroli considered the consolidation of a half-space region by tangential loads applied over a circular region. Gibson et al. have also examined the problem of the axisymmetric and plane strain consolidation of a poroelastic layer underlain by a rigid base. In addition to the traction boundary value problems, a number of researchers have examined mixed boundary value problems related to half space regions of poroelastic media. Such solutions have useful application to the study of consolidation settlement of foundations resting on fluid-saturated cohesive soil media. Heinrich and Desoyer considered the application of Biot's theory to the study of the consolidation settlement of a rigid porous circular foundation in smooth axisymmetric contact with a poroelastic half space. The formulation of the problem is approximate in the sense that the contact stress beneath the rigid foundation was prescribed a priori according to the result given by Boussinesq and Harding and Sneddon. The problem was reexamined by Agbezuge and Deresiewicz and Chiarella and Booker who developed approximate solutions to the integral equations governing the mixed boundary value problem. The problem was also examined by Gaszynski and Szefer, who considered both permeable and impermeable conditions at the indenting surface.

The objective of the paper is to extend these studies to include the class of contact problems involving asymmetric indentation of a poroelastic half space. Such asymmetric indentation can occur as a result of eccentric loading of the indenting foundation (Figure 1). It is assumed that the eccentric loading does not induce separation at the contact surface between the circular
foundation and the poroelastic half space. For separationless contact, and in view of the linearity of the basic equations governing the poroelastic medium, the eccentric loading problem can be reduced to the combination of a purely axisymmetric problem involving central loading of the foundation by a force $P_c$ and the purely asymmetric loading by a moment $M_y = P_c e$.

Because the solution to the first problem is available in the literature, the developments presented in this paper focus purely on the study of the loading due to the moment. Using a combination of Hankel and Laplace transforms, the paper develops the basic integral equations governing the asymmetric indentation of the half space region. In particular, the constitutive assumptions consider compressibility of the pore fluid. In conventional treatments of poroelasticity problems applicable to geomaterials, it is usually assumed that the pore fluid is incompressible (i.e., fully saturated, no pore air voids, etc.). Consequently, the undrained Poisson ratio $v < 0.5$ and Skempton’s pore pressure parameter $B = 1.15$. When the pore fluid is compressible, both $v_c$ and $B$ are governed by the limits $0 < v_c < 0.5$ and $0 < B < 1$, respectively. The solution of the governing integral equations and the Laplace transform inversion are achieved via numerical schemes. The final results evaluate the influence of drainage conditions and pore water compressibility on the rotation of the rigid circular foundation, which is subjected to a moment in the form of a Heaviside step function.

2. Governing equations

The equations governing the linear, isotropic behavior of a poroelastic medium saturated with a compressible pore fluid can be presented in the cylindrical coordinates $(r, \theta, z)$ and the temporal domain $t \in [0^+ , \infty)$. The constitutive relations take the form

$$\sigma_{rr} = \frac{2\mu v}{1 - 2v} (\varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{zz}) + 2\mu \varepsilon_{rr}$$

$$\sigma_{\theta\theta} = \frac{2\mu v}{1 - 2v} (\varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{zz}) + 2\mu \varepsilon_{\theta\theta}$$

$$\sigma_{zz} = \frac{2\mu v}{1 - 2v} (\varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{zz}) + 2\mu \varepsilon_{zz}$$

$$p = \frac{2\mu B^2 (1 - 2v)(1 + v_c) \varepsilon_{rr} + 2\mu B (1 + v_c)}{3(1 - 2v)} \varepsilon_{\theta\theta}$$

In the absence of body forces, the quasistatic equations of equilibrium are

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{z\theta}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0$$

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{z\theta}}{\partial z} + \frac{2\sigma_{r\theta}}{r} = 0$$

$$\frac{\partial \sigma_{z\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{z\theta}}{\partial \theta} + \frac{\partial \varepsilon_{zz}}{\partial z} + \frac{\sigma_{zz}}{r} = 0$$

The strain components are related to the displacement components $u_r$, $u_\theta$, and $u_z$ according to

$$\varepsilon_{rr} = \frac{1}{r} \frac{\partial u_r}{\partial r}, \quad \varepsilon_{\theta\theta} = \frac{1}{r} \frac{\partial u_\theta}{\partial \theta}, \quad \varepsilon_{zz} = \frac{1}{r} \frac{\partial u_z}{\partial z}$$

Darcy’s law for quasistatic fluid flow gives

$$u_r = -\kappa \frac{\partial p}{\partial r}, \quad u_\theta = -\kappa \frac{1}{r} \frac{\partial p}{\partial \theta}, \quad u_z = -\kappa \frac{\partial p}{\partial z}$$

The continuity equation of quasistatic fluid flow is

$$\frac{\partial \varepsilon_{rr}}{\partial t} + \frac{\partial u_r}{\partial r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0$$

The above governing equations are characterized by the five basic material parameters, which are represented by the drained and undrained Poisson ratios $v$ and $v_c$, shear modulus $\mu > 0$, Skempton’s pore pressure coefficient $B$, and $\kappa (= k/\gamma_p > 0$, where $k$ is the coefficient of permeability and $\gamma_p$ is the unit weight of pore fluid). For the definition of positive strain energy, it is shown that the material parameters have the following constraints: $\mu > 0$, $0 \leq B \leq 1$, $-1 < v < v_c \leq 0.5$, and $\kappa > 0$ (see, e.g., Rice and Cleary).
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It has been shown by Yue\textsuperscript{17} that the following sets of solution representations exist for the field variables in a linear, isotropic, poroelastic medium of semi-infinite extent that is saturated with a compressible pore fluid. In both the temporal domain and the Laplace transform domain we have

\[
 u(r, \theta, z, t) = \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\infty} wK_\rho \rho d\rho dp \]

\[
 T_z(r, \theta, z, t) = \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\infty} \Pi_a \tau K_\rho \rho d\rho dp \]

\[
 v(r, \theta, z, t) = \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\infty} \rho \Pi_a \theta K_\rho \rho d\rho dp \]

\[
 p(r, \theta, z, t) = \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\infty} \rho \Pi_a \pi K_\rho \rho d\rho dp \]

where \( 0 \leq z < \infty, \ 0 < t < \infty, \) and the integrals are interpreted in the sense of a Cauchy principal value. The vector fields in (6a) are defined by

\[
 u = \begin{pmatrix} u_r \\ u_\theta \\ u_z \end{pmatrix}, \quad T_z = \begin{pmatrix} \sigma_{rr} \\ \sigma_{r\theta} \\ \sigma_{r\theta} \end{pmatrix}, \quad v = \begin{pmatrix} \tau_r \\ \tau_\theta \\ \tau_z \end{pmatrix}
\]

\[
 w = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix}, \quad \tau = \begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{pmatrix}, \quad g = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix}
\]

and the Fourier matrix kernel functions are

\[
 K = e^{ipr \sin(\theta + \phi)}, \quad \Pi_a = \begin{pmatrix} \rho & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \rho \end{pmatrix} \]

\[
 K* = e^{-ipr \sin(\theta + \phi)}, \quad \Pi_a* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

The governing equations (1)-(5) can be rewritten as two sets of first-order ordinary differential equations in the Fourier and Laplace transform domains, i.e.,

\[
 \frac{d}{dz} U_\sigma(z) = \rho C_\sigma U_\sigma(z), \quad \frac{d}{dz} U_\tau(z) = \rho C_\tau U_\tau(z)
\]

where

\[
 U_\sigma(z) = \begin{pmatrix} \hat{\varphi}_2(z) \\ \hat{\varphi}_7(z) \\ \hat{\varphi}_9(z) \end{pmatrix}, \quad C_\tau = \begin{pmatrix} 0 & 1 \\ \mu & 0 \end{pmatrix}
\]

\[
 U_\tau(z) = \begin{pmatrix} \hat{T}_2(z) \\ \hat{T}_7(z) \\ \hat{T}_9(z) \end{pmatrix}
\]

\[
 C_u = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\frac{1}{\kappa} \end{pmatrix}
\]

and

\[
 \gamma = \sqrt{\frac{1}{c \rho^2 + 1}}, \quad c = \frac{2\mu B^2(1 - v)(1 + v) \kappa}{9(v_u - v)(1 - v_u)},
\]

\[
 \lambda = \frac{2\mu v}{1 - 2v}, \quad \mu = \frac{3(v_u - v)}{B(1 - 2v)(1 - v_u)}
\]

and the superscript * stands for the Laplace transform of the particular function with respect to \( t, \) and \( s \) is the Laplace transform parameter.

An algebraic solution representation of the field variables in the transform domains can be further obtained by solving the ordinary differential equations together with the four regularity conditions required as \( z \to +\infty; \) i.e.,

\[
 U_\sigma(z) = \frac{1}{2} [A_q e^{-\rho z} U_\sigma(0)],
\]

\[
 U_\tau(z) = \frac{1}{2} [Q_q e^{-\rho z} + Q_q e^{-\rho z} + \rho z Q_q e^{-\rho z}] U_\tau(0)
\]

where \( 0 \leq z < \infty, \) and the four coefficient matrices \( A_q, \ Q_q, \ Q_q, \) and \( Q_q \) are defined in Appendix A.

By substituting \( z = 0 \) into the algebraic solution representation (8) we have boundary algebraic equations for the eight variables at the exterior surface of the poroelastic half space. It is found that there are only four independent boundary algebraic equations at the
exterior surface of a poroelastic half space; i.e.,

\[
\left( 1 - \frac{1}{\mu} \right) U_p(0) = 0
\]

\[
\begin{pmatrix}
1 & -1 & \frac{1}{2\mu} & 0 & 0 \\
-\xi & -1 & \frac{1}{2\mu} & \frac{1}{2\mu} & 0 \\
\xi & \frac{1}{\mu} + \xi & \frac{1}{\mu} + \xi & \frac{1}{2\mu} \xi & \frac{1}{2\mu} \xi \\
\lambda_u + \mu & \lambda_u + \mu & \lambda_u + 3\mu & \lambda_u + 3\mu & -1 \xi \\
\lambda_u + 2\mu & \lambda_u + 2\mu & 2\mu(\lambda_u + 2\mu) & 2\mu(\lambda_u + 2\mu) & -1 \xi \\
\end{pmatrix}
\begin{pmatrix}
U_p(0) \\
0 \\
0 \\
0 \\
0 
\end{pmatrix} = 0
\]

(9a)

\[
\lambda_u = \frac{2\mu v_u}{1 - 2v_u},
\]

\[
\xi = \frac{3(v_u - v)}{B(1 + v_u)(\gamma^2 - 1)}
\]

(9b)

3. Rigid circular foundation problems

We consider the problem of a rigid circular foundation in smooth contact with a poroelastic half space saturated with a compressible pore fluid and subjected to a moment resultant \( M_0(t) \) about a diametral axis (say, the \( y \)-axis). The drainage boundary conditions are considered to be either completely permeable, partially permeable, or completely impermeable in order to examine their influence on the consolidation behavior of the rigid circular foundation (see Figure I). The mixed boundary conditions at the surface \( z = 0 \) of the poroelastic half space \( 0 < z < \infty \) can be written as follows:

\[
\sigma_{zz}(r, \theta, 0, t) = 0,
\]

\[
u_z(r, \theta, 0, t) = r\omega(t) \cos \theta,
\]

\[
\sigma_{\theta z}(r, \theta, 0, t) = 0,
\]

\[
\sigma_{\theta \theta}(r, \theta, 0, t) = 0,
\]

and the three types of drainage conditions at the surface of the poroelastic half space are given by the following:

**Case I:** The surface of the poroelastic half space is assumed to be a completely pervious surface both within and exterior to the foundation, i.e.,

\[
p(r, \theta, 0, t) = 0, \quad 0 \leq r < \infty
\]

(10b)

**Case II:** The interface between the foundation and the soil is assumed to be impervious and the exterior region is assumed to be pervious, i.e.,

\[
u_z(r, \theta, 0, t) = 0, \quad 0 \leq r < a
\]

\[
p(r, \theta, 0, t) = 0, \quad a < r < \infty
\]

(10e)

**Case III:** The surface of the poroelastic half space is assumed to be a completely impervious surface both within and exterior to the foundation, i.e.,

\[
u_z(r, \theta, 0, t) = 0, \quad 0 \leq r < \infty
\]

(10d)

where \( 0^+ \leq t < \infty, 0 \leq \theta < 2\pi \) and \( \Omega_p(t) \) is the central rotation of the rigid circular foundation about the \( y \)-axis.

By using the sets of solution representations and the boundary algebraic equations, we can obtain the following relations among the relevant boundary variables in the transform domain:

For **Case I** we have

\[
\hat{\omega}_3(\rho, \phi, 0, s) = a_1(\gamma)\hat{\xi}_3(\rho, \phi, 0, s), \quad \hat{p}_w(\rho, \phi, 0, s) = 0
\]

(11a)

For **Case II** we have

\[
\hat{\omega}_3(\rho, \phi, 0, s) = a_1(\gamma)\hat{\xi}_3(\rho, \phi, 0, s) + a_2(\gamma)\hat{\xi}_3(\rho, \phi, 0, s) \quad \hat{\xi}_3(\rho, \phi, 0, s) = 0
\]

(11b)

And for **Case III** we have

\[
\hat{\omega}_3(\rho, \phi, 0, s) = a_3(\gamma)\hat{\xi}_3(\rho, \phi, 0, s), \quad \hat{\xi}_3(\rho, \phi, 0, s) = 0
\]

(11c)

For all three cases we have the zero shear tractions in the transform domains; i.e.,

\[
\hat{\xi}_3(\rho, \phi, 0, s) = 0, \quad \hat{\xi}_3(\rho, \phi, 0, s) = 0
\]

(11d)

The coefficients in equations (11a), (11b) and (11c) are defined by

\[
a_1(\gamma) = \frac{1}{\sigma_u} \left[ 1 + \frac{x_1}{2} k_1(\gamma) \right],
\]

\[
a_2(\gamma) = \frac{1}{\sigma_u} \frac{3x_1}{4B(1 + v_u)} \left[ 1 + k_1(\gamma) \right], \quad a_3(\gamma) = -\frac{\mu}{1 - v}
\]

(11b)

\[
a_3(\gamma) = \frac{2}{3} \kappa B(1 + v_u) k_2(\gamma), \quad a_4(\gamma) = \kappa [1 + k_2(\gamma)],
\]

(11e)

\[
k_1(\gamma) = \frac{1 - \gamma}{\gamma + 1 - x_1}, \quad k_2(\gamma) = \frac{\gamma^2 - 1}{\gamma + 1 - x_1},
\]

\[
k_3(\gamma) = \frac{2 - \gamma(\gamma + 1)}{\gamma(\gamma + 1) - x_1}, \quad x_1 = \frac{2(v_u - v)}{1 - v}
\]

By using these expressions and the sets of solution representations, we can obtain the two-dimensional Fourier-transform-based integral equations, in the Laplace transform domain, for the rigid circular
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foundation problems associated with the three types of drainage boundary conditions. These systems of 2-D integral equations can be further reduced to the systems of Fredholm integral equations of the second kind. In the ensuing, we present only the derivation and results for Case II due to the similarity of the three cases. The results for Cases I and II are given in Appendix B.

\[ \frac{1}{2\pi} \int_0^{2\pi} \left[ a_1(\gamma) \xi_3(\rho, \varphi, 0, s) + a_2(\gamma) \hat{p}_w(\rho, \varphi, 0, s) \right] \times k d\varphi d\rho = r \chi(s) \cos \theta, \quad 0 \leq r < a \]

\[ \frac{1}{2\pi} \int_0^{2\pi} \left[ a_3(\gamma) \xi_3(\rho, \varphi, 0, s) + a_4(\gamma) \hat{p}_w(\rho, \varphi, 0, s) \right] \times k \rho^2 d\varphi d\rho = 0, \quad 0 \leq r < a \] (12)

where 0 \leq \varphi < 2\pi.

We assume that the unknown variables in the two-dimensional boundary integral equations have the following Fourier series expansions:

\[ \xi_3(\rho, \varphi, 0, s) = \sum_{m=-\infty}^{\infty} \xi_{3m}(\rho, s) e^{-im\varphi}, \]

\[ \hat{p}_w(\rho, \varphi, 0, s) = \sum_{m=-\infty}^{\infty} \hat{p}_{wm}(\rho, s) e^{-im\varphi}, \] (13)

where the integer \( m = 0, \pm 1, \pm 2, \cdots, \pm \infty \).

Substituting the Fourier series expansions (13) into the two-dimensional integral equations (12), we can reduce the resulting equations into infinite sets of one-dimensional Hankel-transform-based integral equations:17

\[ \int_0^{\infty} \left[ a_1(\gamma) \xi_{3m}(\rho, s) + a_2(\gamma) \hat{p}_{wm}(\rho, s) \right] J_m(\rho) d\rho = \begin{cases} r \chi(s)/2, & m = \pm 1, 0 \leq r < a \\ 0, & m \neq \pm 1, 0 \leq r < a \end{cases} \]

\[ \int_0^{\infty} \left[ a_3(\gamma) \xi_{3m}(\rho, s) + a_4(\gamma) \hat{p}_{wm}(\rho, s) \right] J_m(\rho) \rho^2 d\rho = 0, \quad 0 \leq r < a \] (14)

\[ \int_0^{\infty} \xi_{3m}(\rho, s) J_m(\rho) \rho d\rho = 0, \quad a < r < \infty \]

\[ \int_0^{\infty} \hat{p}_{wm}(\rho, s) J_m(\rho) \rho d\rho = 0, \quad a < r < \infty \]

By considering the sets of integral equations (14), we have the symmetric solutions for the two unknown variables \( \xi_{31}(\rho, \varphi, 0, s) \) and \( \hat{p}_{w1}(\rho, 0, \varphi, s) \) associated with Case II; i.e.,

\[ \xi_{31}(\rho, \varphi, 0, s) = -2i \sin \varphi \hat{\xi}_{31}(\rho, s), \]

\[ \hat{p}_{w1}(\rho, \varphi, 0, s) = -2i \sin \varphi \hat{\xi}_{w1}(\rho, s) \] (15)

where the unknown variables \( \xi_{31}(\rho, s) \) and \( \hat{p}_{w1}(\rho, s) \) are governed by the following set of integral equations.

\[ \int_0^{\infty} \left[ 1 + k_1(\gamma) \right] \xi_{31}(\rho, s) + \frac{3\gamma_1}{4B(1 + v_2)} \times \left[ 1 + k_1(\gamma) \right] \hat{p}_{w1}(\rho, s) \right] J_1(\rho) \rho d\rho = \frac{r}{2} \Omega(s), \]

\[ 0 \leq r < a \]

\[ \int_0^{\infty} \left\{ k_2(\gamma) \xi_{31}(\rho, s) + \frac{3\gamma_1}{2B(1 + v_2)} \left[ 1 + k_2(\gamma) \right] \hat{p}_{w1}(\rho, s) \right\} J_1(\rho) \rho^2 d\rho = 0, \quad 0 \leq r < a \]

\[ \int_0^{\infty} \xi_{31}(\rho, s) J_1(\rho) \rho d\rho = 0, \quad a < r < \infty \]

\[ \int_0^{\infty} \hat{p}_{w1}(\rho, s) J_1(\rho) \rho d\rho = 0, \quad a < r < \infty \] (16)

For Cases I, II, and III, we can express the unknown contact stress beneath the rigid disc foundation as follows:

\[ \hat{\sigma}_{zz}(r, \theta, 0, s) = 2 \cos \theta \int_0^{\infty} \xi_{31}(\rho, s) J_1(\rho) \rho d\rho \] (17a)

and the moment resultant \( \hat{M}_{p(r)} \) in the Laplace transform domain has the following relation with \( \xi_{31}(\rho, s) \):

\[ \hat{M}_{p(s)} = -2\pi a^2 \int_0^{\infty} \xi_{31}(\rho, s) J_2(\rho) \rho d\rho \] (17b)

4. Fredholm integral equations governing the settlement of the rigid circular foundation

The set of Hankel transform-based integral equations (16) is one of singular integral equations with unknown singularities (see, e.g., Atkinson18). Further simplifications and reductions have to be made to account for the unknown singularities from the integral equations. By isolating these singularities, it is found that the sets of integral equations can be reduced to the systems of Fredholm integral equations of the second kind in the Laplace transform domain. Such systems of complex Fredholm integral equations of the second kind are standard and regular integral equations and can be numerically evaluated if they are not amenable to solution in exact closed forms (see, e.g., Kauwal19, Atkinson20; Baker21). In this section, we will develop the systems of Fredholm integral equations of the second...
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kind for the central rotation of a rigid circular foundation on a poroelastic medium saturated with a compressible fluid.

The last two equations in equations (16) are automatically satisfied by the following representations of solutions in terms of the auxiliary functions \( \hat{\phi}_1(r, s) \) and \( \hat{\psi}_1(r, s) \) (see, e.g., Sneddon\(^{22}\)):

\[
\begin{align*}
\hat{\xi}_{31}(r, s) &= \int_0^a \hat{\phi}_1(x, s) \sin(\rho x) \, dx \\
\hat{\rho}_{21}(r, s) &= \frac{2 B(1 + \nu_u)}{3 \sqrt{r}} \int_0^a \hat{\psi}_1(x, s) J_{3/2}(\rho x) \, dx 
\end{align*}
\]  
(18)

where \( \lim_{x \to 0} \sqrt{x} \hat{\psi}_1(x, s) = 0 \).

We also note that:

\[
\begin{align*}
\int_0^\infty t_{31}(r, s) J_1(\rho r) \, d\rho &= \frac{1}{r} \int_0^r \frac{x}{\sqrt{r^2 - x^2}} \frac{d\hat{\phi}_1(x, s)}{dx} \, dx \\
\int_0^\infty \hat{\rho}_{21}(r, s) J_1(\rho r) \rho^2 \, d\rho &= \frac{2 B(1 + \nu_u)}{3r} \int_0^a \frac{d}{dx} \left[ \sqrt{x} \hat{\psi}_1(x, s) \right] \frac{dx}{\sqrt{r^2 - x^2}} \\
&\quad \times r J_1(\rho r) \rho^2 \, d\rho = 0 
\end{align*}
\]  
(19)

Then the set of integral equations (16) can be reduced to the integral equations of the Abel type,:

\[
\begin{align*}
\int_0^r \frac{x \hat{\phi}_1(x, s)}{\sqrt{r^2 - x^2}} \, dx + \int_0^\infty \left\{ \frac{3\alpha_1}{2} k_1(y) \hat{\xi}_{31}(r, s) \\
+ \frac{3\alpha_1}{4B(1 + \nu_u)} [1 + k_1(y)] \hat{\rho}_{21}(r, s) \right\} \\
&\quad \times r J_1(\rho r) \, d\rho = \frac{a \rho^2}{2} \hat{\Omega}_1(s) \\
\int_0^\infty \frac{d}{dx} \left[ \sqrt{x} \hat{\psi}_1(x, s) \right] \frac{dx}{\sqrt{r^2 - x^2}} + \int_0^\infty \left\{ k_2(y) \hat{\xi}_{31}(r, s) \\
&\quad + \frac{3}{2B(1 + \nu_u)} k_2(y) \hat{\rho}_{21}(r, s) \right\} r J_1(\rho r) \rho^2 \, d\rho = 0 
\end{align*}
\]  
(20)

By using the following formulas\(^{22}\):

\[
\begin{align*}
g(x) &= \frac{2}{\pi} \int_0^x \frac{f(y)}{\sqrt{x^2 - y^2}} \, dy, \\
f(y) &= \frac{2}{\pi} \int_0^y \frac{x g(x)}{\sqrt{y^2 - x^2}} \, dx 
\end{align*}
\]  
(21)

The solution of the Abel-type integral equations (20) can be written as:

\[
\begin{align*}
\begin{pmatrix} \hat{\phi}_1(r, s) \\ \hat{\psi}_1(r, s) \end{pmatrix} &= \left( \frac{2\pi \alpha}{\Omega_1(s)} \right)^r \begin{pmatrix} K_{11}(r, y, s) & K_{12}(r, y, s) \\ K_{13}(r, y, s) & K_{14}(r, y, s) \end{pmatrix} \begin{pmatrix} \hat{\phi}_1(y, s) \\ \hat{\psi}_1(y, s) \end{pmatrix} \\
&= \left( \frac{2\pi \alpha}{\Omega_1(s)} \right)^r \begin{pmatrix} \hat{\phi}_1(y, s) \\ \hat{\psi}_1(y, s) \end{pmatrix} \\
&= \begin{pmatrix} \hat{\phi}_1(r, s) \\ \hat{\psi}_1(r, s) \end{pmatrix} \\
dy = \left( \frac{2\pi \alpha}{\Omega_1(s)} \right)^r \\
&= 0 
\end{align*}
\]  
(22a)

where \( 0 \leq r < a \); the kernel functions in the integral equations (22a) are given below:

\[
\begin{align*}
K_{11}(r, y, s) &= \frac{\alpha_1}{\pi} \int_0^\infty k_1(y) \sin(\rho r) \sin(\rho y) \, d\rho \\
K_{12}(r, y, s) &= \frac{\alpha_1}{\sqrt{2\pi}} \int_0^\infty \frac{1}{\sqrt{\rho}} \times [1 + k_1(y)] \sin(\rho r) J_{3/2}(\rho y) \, d\rho \\
K_{13}(r, y, s) &= \frac{2}{\pi} \int_0^\infty k_2(y) J_{3/2}(\rho y) \sin(\rho r) \rho^{3/2} \, d\rho \\
K_{14}(r, y, s) &= \frac{\alpha_1}{\pi} \int_0^\infty k_3(y) \sin(\rho r) \sin(\rho y) \, d\rho 
\end{align*}
\]  
(22b)

The unknown contact normal stress in the Laplace transform domain can be expressed in terms of the auxiliary function \( \hat{\phi}_1(r, s) \) for the rigid circular foundation problem associated with the three drainage conditions:

\[
\hat{\sigma}_z(r, \theta, 0, s) = -2 \cos \theta \frac{d}{dr} \int_0^a \hat{\phi}_1(x, s) \, dx 
\]  
(22c)

where \( 0 \leq r < a \).

The associated moment resultant \( \hat{M}_s(s) \) in the Laplace transform domain can be evaluated in the following form:

\[
\hat{M}_s(s) = -4\pi \int_0^a x \hat{\phi}_1(x, s) \, dx 
\]  
(22d)

The equations (22a) are the Fredholm integral equations of the second kind in the Laplace transform domain for the central rotation of a rigid circular foundation on a saturated poroelastic medium associated with the drainage boundary condition of Case II.

By introducing the nondimensional variables \( \hat{\phi}(x, s_1), \hat{\psi}(x, s_1), \hat{X}_s(s_1), \) and \( \hat{Y}_s(s_1) \) as follows:

\[
\begin{align*}
&\hat{x} = \frac{r}{a}, \quad \hat{\phi}_1(r, s) = -\frac{M_y}{4\pi a^2} \hat{\phi}(x, s_1), \\
&\hat{\Omega}_1(s) = \frac{1 - \nu}{8\mu a^3} M_y \hat{X}_s(s_1) \\
&\hat{\psi}_s(r, s) = -\frac{M_y}{4\pi a^2} \sqrt{x s_1} \hat{\psi}(x, s_1), \\
&\hat{M}_s(s) = M_y \hat{Y}_s(s_1) 
\end{align*}
\]  
(23)

we can rewrite the systems of complex Fredholm integral equations of the second kind (22) in the following non-dimensional forms:

\[
\begin{align*}
\begin{pmatrix} \hat{\phi}_1(x, s_1) \\ \hat{\psi}_1(x, s_1) \end{pmatrix} &= \left( \frac{2\pi \alpha}{\hat{\Omega}_1(s_1)} \right)^r \begin{pmatrix} K_{11}(x, y, s_1) & K_{12}(x, y, s_1) \\ K_{13}(x, y, s_1) & K_{14}(x, y, s_1) \end{pmatrix} \begin{pmatrix} \hat{\phi}_1(y, s_1) \\ \hat{\psi}_1(y, s_1) \end{pmatrix} \\
&= \left( \frac{2\pi \alpha}{\hat{\Omega}_1(s_1)} \right)^r \begin{pmatrix} \hat{\phi}_1(y, s_1) \\ \hat{\psi}_1(y, s_1) \end{pmatrix} \\
&= \begin{pmatrix} \hat{\phi}_1(x, s_1) \\ \hat{\psi}_1(x, s_1) \end{pmatrix} \\
&= 0 
\end{align*}
\]  
(24a)

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Also equation (22d) reduces to

\[ \int_0^1 y \phi_1(y, s) dy = \hat{Y}_s(s_1) \]  

(24b)

where \( 0 \leq x \leq 1 \), and the nondimensional forms of the kernel functions are defined by

\[ \hat{K}_1(x, y, s) = -\frac{s_1}{\pi \rho^2} \int_0^{s_1} \frac{s_1 \sin(\rho x) \sin(\rho y)}{(2 - \alpha_x)(\rho^2 + \rho \sqrt{\rho^2 + s_1}) + s_1} d\rho \]

\[ \hat{K}_2(x, y, s) = -\frac{s_1}{\pi \rho^2} \int_0^{s_1} \frac{s_1 \sin(\rho x) \sin(\rho y)}{(2 - \alpha_x)(\rho^2 + \rho \sqrt{\rho^2 + s_1}) + s_1} d\rho \]

\[ \hat{K}_3(x, y, s) = -\frac{s_1}{\pi \rho^2} \int_0^{s_1} \frac{s_1 \sin(\rho x) \sin(\rho y)}{(2 - \alpha_x)(\rho^2 + \rho \sqrt{\rho^2 + s_1}) + s_1} d\rho \]

\[ \hat{K}_4(x, y, s) = \frac{2}{\pi} \int_0^{s_1} \frac{s_1 \sin(\rho x) \sin(\rho y)}{(2 - \alpha_x)(\rho^2 + \rho \sqrt{\rho^2 + s_1}) + s_1} d\rho \]

(24c)

To solve the integral equations numerically, the interval \([0, 1]\) is divided into \( N \) segments with ends defined by \( x_k = (k - 1)/N; k = 1, 2, 3, \ldots (N + 1) \), and the collocation points are \( x_k = (x_{k+1} + x_k)/2 \), \( k = 1, 2, 3 \ldots N \). Consequently we can convert the above integral equations into three systems of linear algebraic equations. These linear algebraic equations can be written in the generalized matrix form

\[ \sum_{j=1}^L A_{ij} [X_j] = [B_i] \]  

(27)

where \( l = 1, 2, 3, \ldots, L \), \( L = 2N + 2 \) for Cases I and III and \( L = 4N + 2 \) for Case II.

The matrix equation (27) can be solved numerically to generate the unknown variables \( \phi(x, s_1) \), \( \psi(x, s_1) \), and \( \hat{X}_s(s_1) \) (or \( \hat{Y}_s(s_1) \)) in the Laplace transform domain. The time-dependent results of the variables can be evaluated by using algorithms for the numerical inversions of Laplace transforms.

The numerical techniques discussed here consist of three computational steps that are essentially based on three-dimensional repeated numerical integrations. The first step involves the numerical integration of the infinite integrals, which occur in the kernel functions (24c). As noted previously, these infinite integrals are absolutely
convergent. However, there are two unusual aspects of the numerical integrations. One refers to the infinite limit and the other is associated with the integrands. As the values of the independent variables $s$ and $p$ become large, the integrands will become rapidly oscillatory functions. This property of the integrands will slow down the convergence of the numerical integration and will result in an unstable numerical integration procedure. The large values of the complex variable $s$ are associated with small values of time $t$ in the numerical inversion of Laplace transforms. A particular technique adopted in this study overcomes these two numerical problems associated with the evaluation of infinite integrals with rapidly oscillatory integrands. This particular technique consists of Simpson's quadrature-based proceeding limit technique and an approximation technique, the details of which are documented in Appendix C. The second repeated numerical integration involves the systems of Fredholm integral equations whose properties have been well investigated by many researchers (e.g., Baker21; Delves and Mohamed22; and Selvadurai et al.23,24). These studies show that with the increase in the segment number $N$ it is possible to obtain more accurate and readily convergent solutions for the integral equations. The last computational step involves the numerical inversion of Laplace transforms. A modified algorithm is proposed and verified for the numerical inversion of Laplace transforms (see Appendix D).

Due to the accumulations of the errors in each repeated numerical integration, however, it is necessary to evaluate and test the convergence and accuracy of the time-dependent results for the systems of Fredholm integral equations of the second kind in the Laplace transform domain. The details of these calculations are given by Yue.17. It is concluded from this verification that the numerical scheme and techniques adopted in this study provide highly stable and accurate solutions in the time domain for the systems of Fredholm integral equations of the second kind in the Laplace transform domain and the procedures particularly overcome the numerical problems customarily associated with the initial stages ($10^{-4} \leq ct/a^2 \leq 10^{-2}$) of the consolidation of the poroelastic medium.

6. Analytical results

It is instructive first examine the results for the three limiting cases that pertain to the initial and final responses by using the Tauberian theorems22 as well as the limiting response of $v \to v_\alpha$. The limiting results for the unknown variables $\phi$ can be found in exact closed form for the central rotation of the rigid circular foundation associated with the three drainage conditions. For Cases I, II, and III we have

\[ \lim_{t \to 0^+} \frac{\phi(x, t)}{X_r(t)} = z_3 \, x, \quad \lim_{t \to \infty} \frac{\phi(x, t)}{X_r(t)} = x, \]
\[ \lim_{r \to \infty} \frac{\phi(x, t)}{X_r(t)} = x \]

(28a)

where $z_3 = (1 - v)/(1 - v_\alpha)$. We then have the limiting results for the relationship between the central rotation $\Omega_f(t)$ and the moment resultant $M_f(t)$, i.e.,

\[ \lim_{t \to 0^+} \frac{M_f(t)}{\Omega_f(t)} = \frac{3(1 - v_\alpha)}{8\mu a^3} \]
\[ \lim_{t \to \infty} \frac{M_f(t)}{\Omega_f(t)} = \frac{3(1 - v)}{8\mu a^3} \]
\[ \lim_{r \to \infty} \frac{M_f(t)}{\Omega_f(t)} = \frac{3(1 - v_\alpha)}{8\mu a^3} \]

(28b)

Second, if the moment resultant is constant with time, i.e., $M_f(t) = M_f(t_f)$, the central rotation $\Omega_f(t)$ for the rigid circular foundation can be expressed as the following nondimensional form:

\[ a^2 \mu \Omega_f(T) - \frac{1 - v}{8} X_f(T) \]

(29a)

The consolidation-induced central rotation of the rigid circular foundation can be expressed as

\[ a^2 \mu \left( \Omega_f(T) - \Omega_f(0) \right) = \frac{3(v_\alpha - v)}{8} U_f(T) \]

(29b)

and the degree of consolidation-induced rotation of the rigid circular foundation can be defined as

\[ U_f(T) = \frac{\Omega_f(T) - \Omega_f(0)\nu \nu (1 + v)}{\Omega_f(\infty) - \Omega_f(0)\nu \nu (1 + v)} = \frac{X_f(T) - 3z_3}{3 - 3z_3} \]

(29c)

where $X_f(T)$ is a nondimensional function evaluated from the integral equations (27). The function $X_f(T)$ includes only the material constants of the drained and undrained values of Poisson's ratios. The nondimensional time factor $T$ is defined by

\[ T = \frac{ct}{a^2} = \frac{2\mu B^2 \nu \nu (1 - v)(1 + v^2)}{9(v_\alpha - v)(1 - v_\alpha)} \]

(30)

From the preceding, we can make the following observations:

1. The role of the material parameters $\mu$, $B$, $\nu$ and the length parameter $a$ on the response of the consolidating soil can be analytically examined by using the nondimensional rotation together with the time factor $T$.
2. By using the time factor $T$, only the drained and undrained Poisson's ratios have an influence on the nondimensional rotation of the rigid circular foundation and the associated degree of consolidation. The initial response of the nondimensional rotation is governed by the undrained Poisson ratio $v_\alpha$. The smaller the value of the undrained Poisson ratio, the larger the initial rotation. The final response of the nondimensional rotation is governed by the drained Poisson ratio $v$. The larger the drained Poisson ratio, the smaller the final rotation. The nondimensional consolidation-induced rotation of the rigid circular foundation is directly proportional to the difference between the undrained and drained Poisson ratios (i.e., $v_\alpha - v$) and reaches its maximum value when $v_\alpha = 0.5$ and $v = 0$ with the constraint of $0 \leq v < v_\alpha \leq 0.5$. This maximum value is one half of the total
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rotation of the rigid circular foundation about the y-axis.

3. The limiting response as \( v \to v_a \) indicates that there is no consolidation-induced rotation for the rigid circular foundation and the rotation is complete upon application of the load. The three limiting displacements are identical to those obtained from classical elastostatics by taking the relevant values of Poisson’s ratios (see, e.g., Rycroft \( ^{26} \); Florence \( ^{27} \); and Selvadurai \( ^{28} \)).

7. Numerical results

In the ensuing, we shall present numerical results for the rotation of the rigid circular foundation about a diametral axis by assuming that the rigid circular foundation is subjected to a basic moment resultant in the form of a Heaviside step function \( M_H(t) \). Seven sets of the drained and undrained Poisson ratios are selected to illustrate the influence of the drainage boundary conditions and the Poisson ratios on the consolidation process and on the magnitude of the nondimensional central rotation. These sets of Poisson’s ratios are

\[
(v, v_a) = (0.49, 0.5), (0.4, 0.5), (0.2, 0.5), (0, 0.5), (0, 0.4), (0, 0.2), (0, 0.01)
\]

The drainage boundary conditions at the surface of the consolidating half space are assumed to correspond to either completely permeable (Case I), or partially permeable (Case II), or completely impermeable (Case III) cases. Among these three drainage boundary conditions, Cases I and III are the limiting cases, which can offer the plausible “extreme estimates” for all the “partial” drainage conditions that can occur at the surface of the consolidating half space where the impermeability surface constraints can extend a finite distance beyond the foundation. Case II is a particular example of such an intermediate drainage condition.

The influence of drainage boundary conditions on the consolidation-induced rotation of the rigid circular foundation in smooth contact with a saturated poroelastic half space can be observed in Figures 2–4 where the degree of consolidation is plotted against the logarithm of the nondimensional time factor \( c t/a^2 \). As is evident, the drainage boundary conditions at the exterior surface have a significant effect on the time and rate of the consolidation-induced rotation. The time for a certain degree of the consolidation-induced rotation associated with the intermediate drainage boundary condition (Case II) is bounded by the lower value associated with the completely permeable drainage boundary condition (Case I) and the upper value associated with the completely impermeable drainage boundary condition (Case III). The behavior of the rigid foundation corresponding to the drainage boundary condition defined in Case II approaches that associated with Case III at the initial stages (\( T < 0.001 \)). For larger values of \( T(1) \), the result for Case II approaches that of Case I.

In Figures 5–7, the degree of consolidation-induced rotation \( U_c(T) \) is plotted against the time factor \( T \) for four sets of Poisson’s ratios. The curves in Figures 5–7 for \( (v, v_a) = (0.49, 0.5), (0.0, 0.2) \), and \( (0.4, 0.4) \) are almost indiscernible from those for \( (v, v_a) = (0, 0.01), (0.4, 0.5), \) and \( (0.2, 0.5) \), respectively. From these results it could be concluded that the time for a certain degree of consolidation-induced rotation is not extremely sensitive to the values of the drained and undrained Poisson
The time for a particular value of the degree of consolidation-induced rotation is extremely sensitive to the values of Poisson's ratios. These figures demonstrate that the time for a particular level of consolidation is reduced as \( v \) approaches \( v_u \) and is almost directly proportional to the factor \( (v_u - v) \).

For all pairs of Poisson's ratios, the degree of consolidation induced rotation \( U_r(T) \) is almost equal to 99.9% for Case I and II and 99.6% for Case III as the time factor \( T \) reaches 10.

By introducing a modified time factor \( T_m = c_{m}t/a^2 = 2\mu B^2 xt/a^2 \) that is independent of Poisson's ratios, we can replott the degree of consolidation-induced rotation (see, e.g., Figures 8–10). From these figures, it is evident that...
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The nondimensional rotations of the rigid circular foundation are also plotted in Figures 11–13, where the seven sets of Poisson’s ratios are used. Again these figures illustrate that the magnitudes of the consolidation rotation are directly proportional to the factor \((v_u - v)\). The rate of consolidation-induced rotation (i.e., the slope of the curves) at any time increases with the increase in \((v_u - v)\) and reaches its maximum value when \(v_u = 0.5\) and \(v = 0\).

8. Conclusion

This paper examines the mathematical modelling of the quasistatic behavior of a rigid circular foundation that is in smooth contact with a poroelastic half space saturated with a compressible pore fluid and subjected to a moment resultant about its diametral axis. It is shown that the associated integral equations can be reduced to systems of Fredholm integral equations of the second kind in the Laplace transform domain. The numerical evaluations of these coupled Fredholm integral equations are used to generate the time-dependent rotation of the foundation, which is subjected to a step-function-type moment resultant. Efficient computational algorithms have been proposed for the numerical solution of the sets of Fredholm integral equations. The numerical results presented in the paper indicate that the drainage boundary conditions at the surface of a saturated poroelastic half space have an important influence on the consolidation-induced rotation of the rigid circular foundation. A further aspect considered in these studies relates to the influence of the undrained and drained compressibility. It is shown that the influence of both \(v\) and \(v_u\) on the consolidation response can be clearly and realistically illustrated by using both the conventional time factor \(ct/a^2\) and a modified time factor \(c_{\mu}/a^2\). In the limit when \(v \to v_u\), the poroelastic medium exhibits only an elastic deformation, no excess pore pressures are developed, and consolidation effects are absent.

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References

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Appendix A: Coefficient matrices in the solution representation

\[
A_q = \begin{pmatrix}
1 & -1 & 0 & \frac{1}{2\mu} & \frac{1}{2\mu} \\
-1 & 1 & \frac{1}{2\mu} & \frac{1}{2\mu} & 0 \\
-1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

\[
Q_v = \begin{pmatrix}
\frac{\lambda + \mu}{\lambda + 2\mu} & \frac{\mu}{\lambda + 2\mu} & 1 & -\frac{(x + 3\mu)}{2\mu(\lambda + 2\mu)} & 0 & 0 \\
\frac{\mu}{\lambda + 2\mu} & 1 & -\frac{(x + 3\mu)}{2\mu(\lambda + 2\mu)} & 0 & 0 & 0 \\
-2\mu & -2\mu & 1 & -1 & 0 & 0 \\
2\mu & 2\mu & -1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

\[
Q_q = \begin{pmatrix}
1 & \frac{\mu}{\lambda + 2\mu} & 0 & -\frac{(x + 3\mu)}{2\mu(\lambda + 2\mu)} & 0 & 0 \\
\frac{\mu}{\lambda + 2\mu} & 1 & -\frac{(x + 3\mu)}{2\mu(\lambda + 2\mu)} & 0 & 0 & 0 \\
0 & -2\mu & \frac{\lambda + \mu}{\lambda + 2\mu} & 1 & -\frac{\mu}{\lambda + 2\mu} & 0 \\
-2\mu & \frac{\lambda + \mu}{\lambda + 2\mu} & 0 & -\frac{\mu}{\lambda + 2\mu} & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

(A1)
Appendix B: Integral Equations for Cases I and III

The systems of 2-D Fourier-transform-based integral equations can be expressed by the following for the central rotation of a rigid circular foundation associated with the drainage conditions of Cases I and III.

\[ \begin{align*}
\frac{1}{2\pi} & \int_{0}^{2\pi} \int_{0}^{\pi} b(y)\tilde{\theta}_3(\rho, \phi, 0, s)K_0d\rho d\phi = r\tilde{\Omega}_f(s)\cos \theta, \\
\frac{1}{2\pi} & \int_{0}^{2\pi} \int_{0}^{\pi} \tilde{\theta}_3(\rho, \phi, 0, s)K_0d\rho d\phi = 0, \quad a < r < \infty
\end{align*} \]

where 0 \leq \theta < 2\pi; for Case I, b(\gamma) = a_1(\gamma); and for Case III, b(\gamma) = a_3(\gamma).

The above 2-D integral equations can be reduced to the following sets of Hankel-transform-based integral equations.

\[ \begin{align*}
\int_{0}^{\infty} \left[ 1 + \frac{1}{2} k(\gamma) \right] \xi_3(\rho, s)J_1(\rho) d\rho & = \frac{r}{2} \tilde{\Omega}_f(s), \\
\int_{0}^{\infty} \xi_3(\rho, s)J_1(\rho) d\rho & = 0, \quad a < r < \infty
\end{align*} \]

where for Case I, \( k(\gamma) = k_1(\gamma) \); and for Case III, \( k(\gamma) = k_3(\gamma) \).

The Fredholm integral equations of the second kind in the Laplace transform domain can be found by using the auxiliary function \( \hat{\phi}_1(r, s) \) in equations (20).

\[ \hat{\phi}_1(r, s) + \int_{0}^{a} K_1(r, y, s)\hat{\phi}_1(y, s) dy = \frac{2}{\pi} \tilde{\Omega}_f(s)r \]

where 0 \leq r < a; for Case I, \( K_1(r, y, s) = K_{11}(r, y, s) \); and for Case III, \( K_1(r, y, s) = K_{13}(r, y, s) \).

Using equations (25), we reduce the above equations into the following nondimensional forms, i.e.,

\[ \hat{\phi}_1(x, s_1) + \int_{0}^{1} K(x, y, s_1)\hat{\phi}_1(y, s_1) dy = x\tilde{\Omega}_f(s_1) \]

where 0 \leq x < 1; for Case I, \( K(x, y, s) = \tilde{K}_1(x, y, s) \); and for Case III, \( \tilde{K}(x, y, s) = \tilde{K}_3(x, y, s) \).

The above equations can be further expressed as
follows by using the real variables defined in equations (27) and $\hat{K}(x, y) = \hat{K}_1(x, y) + i\hat{K}_2(x, y)$.

\[
\int_0^1 \left( \hat{K}_1(x, y) - \hat{K}_2(x, y) \right) \phi_1(y) \, dy + \left( \phi_1(x) \right) = \left( \hat{K}_1(x, y) \right) \phi_2(y) \, dy
\]

(B5)

Appendix C: Evaluation of the infinite integrals in the kernel functions

In general, the infinite integrals in the kernel functions can be written in terms of the three independent variables $x, y,$ and $s$; i.e.,

\[
I_1(x, y, s) = \frac{2}{\pi} \int_0^\infty f_1(\rho, s_1) \sin(\rho x) \sin(\rho y) \, d\rho
\]

\[
I_2(x, y, s) = \frac{2}{\pi} \int_0^\infty \frac{1}{\rho} f_1(\rho, s_1) \sin(\rho x) \left[ \frac{\sin(\rho y)}{\rho y} \right] \rho y \, d\rho
\]

\[
I_3(x, y, s) = \frac{2}{\pi} \int_0^\infty f_2(\rho, s_1) \left[ \frac{\sin(\rho x)}{\rho x} - \cos(\rho x) \right] \sin(\rho y) \, d\rho
\]

\[
I_4(x, y, s) = \frac{2}{\pi} \int_0^\infty f_3(\rho, s_1) \sin(\rho x) \sin(\rho y) \, d\rho
\]

We can then rewrite the infinite integrals (C1) in the following forms:

\[
I_1(x, y, s) = \frac{2}{\pi} \int_0^\infty \left[ f_1(\rho, s_1) - c_1(\rho, s_1) \right] \sin(\rho x) \sin(\rho y) \, d\rho + I_{1,0}(x, y, s)
\]

\[
I_2(x, y, s) = \frac{2}{\pi} \int_0^\infty \frac{1}{\rho} \left[ f_1(\rho, s_1) - c_1(\rho, s_1) \right] \sin(\rho x) \left[ \frac{\sin(\rho y)}{\rho y} \right] \rho y \, d\rho + I_{2,0}(x, y, s)
\]

\[
I_3(x, y, s) = \frac{2}{\pi} \int_0^\infty \left[ f_2(\rho, s_1) - c_2(\rho, s_1) \right] \left[ \frac{\sin(\rho x)}{\rho x} - \cos(\rho x) \right] \sin(\rho y) \, d\rho + I_{3,0}(x, y, s)
\]

\[
I_4(x, y, s) = \frac{2}{\pi} \int_0^\infty \left[ f_3(\rho, s_1) - c_3(\rho, s_1) \right] \sin(\rho x) \sin(\rho y) \, d\rho + I_{4,0}(x, y, s)
\]

where $0 \leq x, y < 1, s = s + i\delta, s > 0, -\infty < s_1 < \infty, i = \sqrt{-1}$, and the functions $f_j(\rho, s)(j = 1, 2, 3)$ are

\[
f_1(\rho, s_1) = \frac{s_1}{(2 - \alpha_1)\rho(\rho + \sqrt{\rho^2 + s_1} + s_1)}
\]

\[
f_2(\rho, s_1) = \frac{s_1}{\rho(\sqrt{\rho^2 + s_1} + (1 - \alpha_1)\rho)}
\]

\[
f_3(\rho, s_1) = \frac{s_1 + \rho(\sqrt{s_1^2 + s_1} - \rho^2)}{\rho(1 - \alpha_1)\rho + \sqrt{s_1^2 + s_1} + s_1}
\]

The functions $f_j(\rho, s)(j = 1, 2, 3)$ have the following approximations as $|s|/\rho^2 \to 0$:

\[
f_1(\rho, s) \sim c_1(\rho, s) = \frac{1}{2(2 - \alpha_1)\rho^2} - \frac{4 - \alpha_1}{4(2 - \alpha_1)s_1}
\]

\[
f_2(\rho, s) \sim c_2(\rho, s) = \frac{1}{2 - \alpha_1} - \frac{1}{2(2 - \alpha_1)s_1}
\]

\[
f_3(\rho, s) \sim c_3(\rho, s) = \frac{3}{2(2 - \alpha_1)\rho^2} - \frac{20 - \alpha_1}{12(2 - \alpha_1)s_1}
\]

We can then rewrite the infinite integrals (C4) in the following forms:

\[
I_{1,0}(x, y, s) = \frac{2}{\pi} \int_0^\infty c_1(\rho, s) \sin(\rho x) \sin(\rho y) \, d\rho
\]

\[
I_{2,0}(x, y, s) = \frac{2}{\pi} \int_0^\infty \frac{1}{\rho} c_1(\rho, s) \sin(\rho x) \left[ \frac{\sin(\rho y)}{\rho y} \right] \rho y \, d\rho
\]

\[
I_{3,0}(x, y, s) = \frac{2}{\pi} \int_0^\infty c_2(\rho, s) \left[ \frac{\sin(\rho x)}{\rho x} - \cos(\rho x) \right] \sin(\rho y) \, d\rho
\]

\[
I_{4,0}(x, y, s) = \frac{2}{\pi} \int_0^\infty c_3(\rho, s) \sin(\rho x) \sin(\rho y) \, d\rho
\]

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The infinite integrals in (C5) are integrated in exact closed form with the help of the following results, i.e.,

\[ \int_0^\infty \frac{s \sin(px)}{p^2 + \chi^2} \, dp = \frac{1}{2} \sqrt{s} - \frac{1}{2} \sqrt{s} e^{-s} + \frac{1}{2} \sqrt{s} e^{-s}, \]

and

\[ \int_0^\infty \frac{\sin(px)}{p^2 + \chi^2} \, dp = \frac{1}{2} \sqrt{s} - \frac{1}{2} \sqrt{s} e^{-s} + \frac{1}{2} \sqrt{s} e^{-s}, \]

where \( 0 \leq y \leq x \) and \( \chi > 0 \).

The adjusted infinite integrals in (C4) can be evaluated by using the proceeding limit technique, i.e.,

\[ \int_0^\infty F(p, x, y, s) \, dp \approx \int_0^{A_0} F(p, x, y, s) \, dp + \int_{A_0}^{A_1} F(p, x, y, s) \, dp + \cdots + \int_{A_n}^{A_{n+1}} F(p, x, y, s) \, dp \]  

where \( 0 < A_0 < A_1 < \cdots < A_{n+1} \) is a sequence of numbers that approaches infinity. Each finite integral on the right-hand side is proper and can be calculated by using the Simpson's quadrature-based adaptively iterative integration. The limits of \( A_0, A_1, \ldots, A_{n+1} \) are chosen as the zeros of the oscillatory part of the integrands, i.e., either \( \sin(p(x \pm y)) \) or \( \cos(p(x \pm y)) \). The evaluation of these proceeding finite integrals is automatically terminated provided the following condition is satisfied:

\[ \left| \int_{A_n}^{A_{n+1}} F(p) \, dp \right| \leq \varepsilon_c \]  

where \( \varepsilon_c \) is an assigned absolute error.

As a result, the adjusted infinite integrals in equations (C4) can be evaluated with greater accuracy and faster convergence by using the Simpson's quadrature-based proceeding limit technique. With the above procedure, we can accommodate the problems associated with infinite integrals that contain rapidly oscillatory integrands.

Appendix D: Algorithm for the numerical inversion of Laplace transforms

Let \( f(t) \) be a real function of \( t \), with \( f(t) = 0 \) for \( t < 0 \); the Laplace transform and its inversion formula are defined as follows:

\[ f(s) = \int_0^\infty f(t) e^{-st} \, dt, \quad f(t) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} f(s) e^{st} \, ds \]  

where \( s = \sigma + i\omega \), \( \sigma > \sigma_0 > 0 \), \( \lim_{t \to \infty} f(t) = 0(e^{\sigma t}) \), and \( 0 \) is a constant that exceeds the real part of all the singularities of \( f(s) \), and the integrals are in the sense of the Cauchy principal value. In the ensuing discussion, it is therefore assumed that equations (D1) are defined for \( \text{Re}(s) > a > 0 \).

The following algorithm for numerical inversion of Laplace transforms was proposed by Crump.29 The basic formula for approximating the inverse Laplace transform was derived by expanding the integral in the form of a Fourier series, i.e.,

\[ f(t) \approx \frac{1}{T} \int_{-\infty}^{\infty} \hat{f}(a) + \sum_{n=1}^{\infty} \hat{f}(a + i \frac{n\pi T}{\varepsilon_m}) e^{i\phi} \]  

where \( 0 < t \leq 2T, \phi = \pi t/T \).

The epsilon technique suggested by MacDonald30 was incorporated in the algorithm for a faster convergent approximation of the infinite summation in (D2), i.e.,

\[ \varepsilon_n = \min\{\text{Re}(s_n), \text{Re}(s_m)\}, \quad m = 1, 2, 3, \ldots, 2M + 1 \]

where \( \varepsilon_0 = 0, \quad \varepsilon_1 = \varepsilon_0 + 1, \quad \varepsilon_m = \varepsilon_{m-1} + \frac{1}{\varepsilon_{m-1} - \varepsilon_0}, \quad m = 1, 2, 3, \ldots, 2M + 1 \)

The constant parameter \( a \) in equation (D2) was obtained from error analysis as follows,

\[ a = \frac{\ln(\varepsilon_0)}{2T} \]  

where \( \varepsilon_0 \) is a relative error and \( \varepsilon_0 \) is a number larger than \( \max\{\text{Re}(P)\} \), where \( P \) is a pole of \( f(s) \).

In order to apply the above algorithm to the study of
poroelastic media, we made two modifications as follows. The first modification is implemented to reduce the round-off error induced in the direct calculation of the partial sums of the Fourier series in Crump’s algorithm.

By defining

\[ b_{n+1} = b_{n+2} = 0, \]

\[ b_n = \frac{1}{2}(a + i \frac{n\pi}{T}) + 2 \cos \phi b_{n+1} - b_{n+2}, \]

\[ n = m, m-1, \ldots, 1 \quad (D6) \]

the partial sums can be more accurately evaluated by

\[ S_m = Re[b_1(\cos \phi + i \sin \phi) - b_2] \quad (D7) \]

The second modification is suggested for overcoming the Gibbs phenomena in Crump’s algorithm. Instead of increasing the total number \(2M + 1\) of the partial sums for convergence, we fix the total number \(2M + 1\) and use a time-dependent \(T\) as \(T = \alpha_t\), where \(\alpha\) is a constant. Many functions such as the Heaviside step function, the exponential function, and the error function have been used to verify the accuracy of the algorithm for inverting Laplace transforms. In this study, it was observed that by choosing the constant parameters as \(c_1 = 10^{-6} \sim 10^{-8}\), \(M = 5 \sim 15\), \(\alpha = 0.9 \sim 1.95\), \(\alpha_t = 0\), the adopted algorithm gives optimal results in the time interval of \(10^{-5} \leq t \leq 10^3\). This range of nondimensional time factors meets the requirements applicable to the study of poroelastic media.