On adhesion energy estimates derived from the pressurized brittle delamination of a flexible thin film

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Abstract

This paper examines the problem of the development of a circular delamination due to pressurization at the interface of a flexible elastic film bonded to an elastic substrate. The analysis takes into consideration the elastic deformations of the thin film that can include pure flexure, flexure with shear deformations or large deflections and the elasticity of the substrate. The paper presents explicit results that could be used to estimate either the adhesion energy of the interface or the size of a blister created during the pressurization.

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1. Introduction

The study of the mechanics of delamination of a flexible layer bonded to an elastic substrate has applications to a variety of problem areas in materials science and engineering [1–4]. Bonded flexible coatings are often used to enhance the load carrying capacity of an otherwise softer substrate. In this regard, the integrity of the bond between the reinforcing surface layer and the underlying substrate is of critical importance to the functional use of the composite element. The assessment of adhesive strength, interfacial fracture energy or adhesion energy between a reinforcing surface layer and the underlying substrate has been investigated by many researchers using different experimental techniques and theoretical interpretations. The procedures include interface pressurization [1,2], scratch testing [5,6], four-point bending [7], indentation [8–10] and stressed overlayers [11,12]. In recent studies [13–15], telephone-cord-shaped buckles have also been used to determine the adhesion energies in thin films.

This paper focuses on the modelling of the problem of adhesion energy estimation through the use of an interface pressurization technique. The modelling adopted in many of the studies in this area focuses on the adhesion of two regions consisting of a thin elastic film and a substrate, representing the thin film as a flexible elastic plate that experiences either small strain flexure or small deflection buckling, while treating the substrate as either a rigid base or a deformable elastic region of semi-infinite extent. The elementary modelling has limitations when it is applied to examining thin layer delamination problems where shear deformations and large deflections of the thin layer and elastic deformations of the substrate can exert an influence on the mechanics of the delamination problem. This can lead to errors in interpretation of the interface adhesion energy. This paper therefore examines the problem of the delamination of an interface between an elastic thin film and an elastic substrate by interface pressurization and develops theoretical estimates for the interface adhesion energy through modelling that accommodates flexural and shear energy of the thin film, large deflections of the thin film and elastic energy associated with the deformations of the substrate.

2. Modelling

The problem is considered of an elastic thin film of large extent that is bonded to an elastic substrate with a depth
substantially larger than the thickness \( h \) of the film. A region of the interface between the thin film and the substrate is subjected to pressurization, causing the development of a stable delaminated region of radius \( a \). The deformed configuration of the thin film and the elastic substrate is shown in Fig. 1. In general, the undetached region of the interface will exhibit displacements that satisfy the overall kinematic constraints and the global equilibrium conditions applicable to the problem. In this study, however, it is explicitly assumed that the interface beyond the delaminated zone exhibits zero displacements in the axial direction and furthermore that either no shear tractions develop in the undetached zone or the radial displacements are zero over the entire interface. The assumption related to zero shear tractions implies that the adhesive energy is largely generated by continuity of normal displacements at the interface. This assumption can be modified to include the development of shear stresses only within the unbonded region, but this would entail additional analysis that will not be consistent with the simplifications considered in the overall analysis of the problem presented here, particularly as it relates to the representation of the thin film as a structural element. The analysis of the problem considers that the elastic energy of the thin film is consistent with its deformation behaviour, the elastic energy of the substrate derived through the deformations that occur within the pressurized region and the adhesion energy of the delaminated zone. The objective of the analytical modelling is to develop a relationship that can be used to predict the radius of a stable debonded zone \( a \) for a given pressure \( p_0 \), knowing the adhesion energy \( G \), the thickness of the thin film \( h \) and the elasticity characteristics of the layer and the substrate or, alternatively, to use the relationships to determine the adhesive energy of the interface through a knowledge of the applied pressure and the dimensions of a stable delamination. A Griffith-type brittle delamination criterion is adopted to examine the elastic delamination problem. In accordance with Griffith’s theory, the delamination will propagate when the decrease in the elastic energy of the system balances the energy needed to create the new surfaces of the delamination zone. In order to determine the strain energy released as the delamination extends, the elastic energy of the deformed thin film and the elastic energy of the deformed substrate within the delamination zone are examined.

### 2.1. Elastic energy of the substrate

To estimate the elastic energy of the substrate, the elasticity problem related to a half space region \( (z \geq 0) \) that is subjected to the following mixed boundary value problem is considered:

\[
\sigma_{zz}(r, 0) = -p_0; \quad 0 < r < a
\]

\[
u_z(r, 0) = 0; \quad a \leq r < \infty
\]

\[
u_r(r, 0) = 0; \quad 0 < r < \infty
\]

where \( \nu_z \) is the component of the displacement vector \( \nu \) in the axial direction \( z \) and referred to the cylindrical polar coordinate system \((r, \theta, z)\), and \( \sigma_{zz} \) is the Cauchy stress referred to the same system. The displacement boundary condition (Eq. (2)) and the traction boundary condition (3) are approximations to the conditions applicable to the bonded region \( r \geq a \) beyond the delamination zone. An alternative to the traction boundary condition (Eq. (3)) is to impose a zero displacement boundary condition over the entire interface, i.e.,

\[
u_z(r, 0) = 0; \quad 0 < r < \infty
\]

The specification of a more precise set of displacement and traction conditions is perhaps unwarranted in view of the approximations that will be invoked in modelling the thin film as an elastic structural element. The solution of the mixed boundary value problem defined by Eqs. (1)–(3) is standard and can be found in texts on integral equations, elasticity and fracture mechanics [16,17]. The result of interest here is the deflection of the surface of the substrate in the delaminated zone, which can be evaluated in the generalized form

\[
u_z(r, 0) = \frac{2\zeta(1 - \nu)p_0}{\pi\mu} (a^2 - r^2)^{1/2}
\]

where \( \zeta = 1 \), for the boundary condition for the zero shear traction prescribed at the interface, defined by Eq. (3), \( \zeta = (3 - 4\nu)/4(1 - \nu)^2 \), for the zero radial displacement constraint, prescribed by Eq. (4), and \( \nu \) and \( \mu \) are, respectively, Poisson’s ratio and the linear elastic shear modulus of the substrate material. It may be noted that, when \( \nu = 1/2, \zeta \equiv 1 \), and there is no difference between the results obtained either by imposing the traction boundary condition (Eq. (3)) or the displacement boundary condition (Eq. (4)). The elastic energy released owing to the creation of the delamination zone in the substrate region is given by

\[
U_{\text{Substrate}} = \frac{16\zeta p_0^2 a^3 (1 - \nu)}{3\mu}
\]

### 2.2. Elastic energy of the thin film—a thin plate model

The mechanical behaviour of the thin film can be modelled using a variety of theories in structural mechanics that
take into account mechanical responses, including flexural deformations, shear deformations, membrane effects, etc. The choice of a particular theory of structural mechanics applicable to a thin film cannot be decided at the outset. For example, when the dimension of the delamination zone is much smaller than the thickness of the thin film, it is entirely inappropriate to model the mechanics of the thin film as a structural element; here the mechanical behaviour of the thin film has to be modelled as an elastic solid, which exhibits an axisymmetric state of three-dimensional deformation. In this paper, attention is restricted to three types of thin films: The first category considers thin films with delamination zones for which \(h/a \ll 1\), and the thin film experiences flexure due to the application of the pressure \(p_0\). It is further assumed that the deflections of the thin film are primarily due to flexural deflections that are satisfied by the Poisson–Kirchhoff–Germain thin plate theory [17,18], i.e.,

\[
D\nabla^2\nabla^2 w(r) = p(r)
\]

(7)

where \(D = \mu h^3/(6(1 - v_i))\) is the flexural rigidity of the plate, \(\mu\) and \(v_i\) are, respectively, the linear elastic shear modulus and Poisson’s ratio of the thin film, \(h\) is the thickness of the thin film, \(p(r)\) is an arbitrary pressure and

\[
\nabla^2 = \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr}
\]

is Laplace’s operator for radial symmetry. Since it is assumed that the thin film does not experience any flexure beyond the delamination zone, the relevant kinematic boundary conditions are

\[
w(a) = \left[\frac{dw}{dr}\right]_{r=a} = 0
\]

(9)

For the thin film subjected to a uniform pressure \(p_0\), the solution of Eq. (7) which satisfies the boundary conditions (9) takes the form

\[
w(r) = \frac{p_0}{64D}(a^2 - r^2)^2
\]

(10)

The elastic energy associated with the pure flexural deformations of the thin film is given by

\[
U_{TF}^k = \pi D \int_0^a \left(\nabla^2 w(r)\right)^2 \frac{2(1 - v_p)}{r} \frac{dw(r) d^2w(r)}{dr^2} dr = \frac{\pi a^5 p_0^2}{384D}
\]

(11)

2.3. Elastic energy of the thin film—a thick plate model

In the second category, it is assumed that the thin film also exhibits deflections due to shear. An additional correction to the deflection can be incorporated through either the consideration of shear strains in the thin film or through the use of a higher-order plate theory. The shear stresses through the thickness of the film vary in a parabolic fashion. It is assumed that the mid-surface of the flexible thin film does not stretch or contract, and there are no through-thickness deformations in the thin film. The flexural moments \(M_{rr}, M_{\theta\theta}\) and shear force \(Q_r\) in the thin film can be expressed in terms of the rotation of a plane normal to the mid-surface, \(\psi\). The relevant expressions are

\[
\begin{align*}
M_{rr} &= D \left(\frac{d^2w}{dr^2} + \frac{v_i}{r} \frac{dw}{dr}\right) \\
M_{\theta\theta} &= D \left(\frac{d^2w}{dr^2} + \frac{1}{r} \frac{dw}{dr}\right) \\
Q_r &= \frac{dM_{rr}}{dr} + \frac{(M_{rr} - M_{\theta\theta})}{r}
\end{align*}
\]

(12)

The differential equations governing the rotation \(\psi\) and the displacement \(w(r)\) are given by

\[
D \frac{d}{dr} \left[\nabla^2 (r\psi)\right] = -rp(r); \quad \frac{dw}{dr} = \frac{D}{\kappa^2 \mu h} \left[\frac{1}{r} \nabla^2 (r\psi)\right] - \psi
\]

(13)

where \(p(r)\) is the transverse pressure. The deflection of the thin film can be obtained from relationship (13), where \(\kappa\) is a shear coefficient that depends on the distribution of shear stresses over the thickness of the thin film and on Poisson’s ratio of the material. In the case of the pressurized delamination problem, the applied pressure \(p(r) = p_0\), and the boundary conditions governing the rotation \(\psi\) and the deflection \(w\) are

\[
\psi(a) = 0; \quad w(a) = 0
\]

(14)

In addition, the displacement and rotation fields should be finite and bounded in the domain \(r \in (0, a)\). It can be shown that the rotation and the displacement take the forms

\[
\begin{align*}
\psi &= \frac{p_0}{16D}(a^2 - r^2); \\
w(r) &= \frac{p_0(a^2 - r^2)}{64D} \left[(a^2 - r^2) + \frac{8h^2}{3\kappa^2(1 - v_i)}\right]
\end{align*}
\]

(15)

In the case of the exact solution [18], the shear stress distribution is parabolic and the shear coefficient \(\kappa^2 \approx 2/3\). This value can be compared with other estimates obtained from versions of plate theories that account for shear deformations; for example, the theory proposed by Reissner [19] gives a value of \(\kappa^2 \approx 5/6\) and, for the theory proposed by Mindlin [20], the value of \(\kappa^2\) is determined from the solution of the characteristic equation

\[
4\sqrt{(1 - \zeta \kappa^2)(1 - \kappa^2)} = (2 - \kappa^2)^2
\]

(16)

with \(\zeta = (1 - 2v_i)/2(1 - v_i)\). It may be noted that, when \(0 < v_i < 1/2\), \(0.76 < \kappa^2 < 0.91\) and an approximate average estimate corresponds to \(\pi^2/12\). The elastic energy associated with the deformations of a thin film that experiences flexure and shear deformations is given by

\[
U_{TF}^{FS} = \frac{\pi a^5 p_0^2}{384D} \left[1 + \frac{4}{\kappa^2(1 - v_i)} \frac{h^2}{a^2}\right]
\]

(17)

2.4. Elastic energy of the thin film—a large deflection plate model

The preceding estimates of the elastic energy of the thin film were derived on the basis of the assumption of a thin
plate approximation where the deflections of the thin film were smaller than the film thickness and the corresponding strains in the thin film satisfied Hooke’s law for linear elastic behaviour. Another possible mode of deformation is that, during delamination of the interface through internal pressurization, the thin film experiences large deflections of sufficient magnitude to initiate stretching of the thin film. In the third category, the flexure of the thin film can be examined by considering the stretching displacements \( u(r) \) of the mid-plane of the thin film in addition to its flexural deflection \( w(r) \). There are a number of procedures that can be employed to obtain the solution to the problem of the deflection of a plate that exhibits both flexure and in-plane extension due to large deflection effects. Some of these are given in Refs. \([18,21]\). Here, an approximate analysis is adopted based on a variational procedure. For axisymmetric large deflections of the thin film, the flexural deflections and the stretching deformations are approximated, respectively, by relationships

\[
w(r) = C_0 \left( 1 - \left( \frac{r}{a} \right)^2 \right) ; \quad u(r) = a \left( \frac{r}{a} \right)^2 \left( C_1 + C_2 \frac{r}{a} \right)
\]

where \( C_0, C_1 \) and \( C_2 \) are arbitrary constants. The assumed forms of the deflection \( w(r) \) and the stretching of the mid-plane \( u(r) \) satisfy the condition of flexural fixity at the boundary \( r = a \) and the absence of radial displacements on both the axis of symmetry of the thin film and at the boundary \( r = a \). The variational procedure can be used to develop a result for the energy of the thin film that experiences both flexural deflections and in-plane extension consistent with the assumed forms of the deformations given by Eq. (18). The details of the analysis of the large deflection problem are given in Appendix A. This provides the following estimate for the total strain energy of the thin film undergoing large deflections involving both flexure and stretching:

\[
U_{LD}^{TF} = \frac{\pi p_0^2 a^6}{384D} \left( 1 + \Psi \right)
\]

where the value of \( \Psi \) can be deduced from the results in Eqs. (11), (17) and (19) applicable to the thin film that exhibits pure flexure, flexure with shear deformations or large deflections with in-plane extension, respectively. It may be noted that, as the secondary effects vanish, \( \Psi \to 0 \) and Eqs. (17) and (19) reduce to the classical result in Eq. (11).

Fig. 2. Variation in the normalized surface energy for an elastic thin film–elastic substrate system with modular ratio \( \mu / \mu \) and thickness ratio \( h/a \). Thin film with shear deformation effects.
2.5. Adhesion energy of the delamination

The adhesion energy of the delaminated surfaces created can be calculated by considering the specific adhesion energy $C$ and the areas that are created through the delamination process. (From an experimental point of view, it may be necessary to apply the pressurization over a small radius $b$ in order to create the delamination of a larger radius $a$.) In the general case, the adhesion energy of the delamination is given by

$$ U_S = 2\pi(a^2 - b^2)\Gamma $$

(21)

Any change in the total energy of the system that occurs as a result of the development of a delamination of radius $a$ is given by

$$ \Delta U = \frac{\pi a^6 p_0^2}{384D} (1 + \Psi) + \frac{16\kappa^2 a^3(1 - \nu)}{3\mu} - 2\pi(a^2 - b^2)\Gamma $$

(22)

In terms of the brittle delamination criterion for the interface, the delamination will start to extend when

$$ \frac{\partial}{\partial a} (\Delta U) = 0 $$

(23)

This condition can be used to determine the surface energy associated with the interface delamination, i.e.,

$$ \Gamma = \frac{p_0^2 a}{\mu_i} \left\{ \frac{4\pi(1 - \nu)}{\pi} \left(\frac{\mu_i}{\mu}\right) + \frac{3(1 - \nu)}{128\eta^3} \left\{ 1 + \Omega \left(\frac{p_0}{\mu_i}\right) \right\} \right\} $$

(24)

where

$$ \Omega \left(\frac{p_0}{\mu_i}\right) = \left\{ \begin{array}{ll}
0 & \text{if } p_0/\mu_i < 0.000005 \\
4\kappa^2/\eta(1 - \nu_i) & \text{if } p_0/\mu_i < 0.000005 \\
(21\chi(1 - \nu_i)^2/1024\eta^8)(p_0/\mu_i)^2 & \text{if } p_0/\mu_i \geq 0.000005 \\
\end{array} \right. $$

(25)

depending upon whether the thin film exhibits, respectively, pure flexure, flexure with shear deformation or flexure with in-plane extension.

3. Numerical results

Since the result in Eq. (24) has an explicit analytical form, calculations can be carried out to estimate the three

Fig. 3. Variation in the normalized surface energy for an elastic thin film–elastic substrate system with modular ratio $\frac{\mu}{\mu_i}$, thickness ratio $\frac{h}{a}$ and normalized stress $p_0/\mu_i$. Thin flexible film with effects of in-plane deformations ($\nu = \nu_i = 0$).
possible values of the adhesion energy \( \Gamma \) associated with the three models. The result in Eq. (24) can be used more effectively by examining the values for the surface energy estimates that are influenced by shear effects and in-plane stretching in relation to the surface energy estimates that are derived from the case where the thin film exhibits pure flexure according to the Poisson–Kirchhoff–Germain thin plate theory. This indirectly establishes the conditions under which the simpler pure flexure analysis can be used to estimate the adhesion energy for the interface during debonding under pressurization. Two relative measures are established: for the case where the thin film exhibits flexure and shear deformations:

\[
\Gamma_{FS}^R = 1 + \frac{8\pi \eta^2}{\kappa^2 [3(1 - \nu_f) + 512 \eta^3(1 - \nu_f) \eta^4(\mu_f / \mu)]}
\]

and for the case where the thin film exhibits flexure and in-plane extension:

\[
\Gamma_{FL}^R = 1 + \frac{63\pi \eta^3 (1 - \nu_f)^3}{1024 \eta^3 [3(1 - \nu_f) + 512 \eta^3 (1 - \nu_f) \eta^4(\mu_f / \mu)]} \left( \frac{p_0}{\mu_f} \right)^2
\]

It should be noted that, if the elasticity constants for both the thin film \((\mu_f, \nu_f)\) and the substrate \((\mu, \nu)\) satisfy the thermodynamic constraints \(\mu > 0, \mu_f > 0, -1 \leq \nu < 1/2, -1 \leq \nu_f \leq 1/2\), the second term on the right-hand side of both Eqs. (26) and (27) is positive definite. From this it is concluded that the estimate for the surface energy derived from the thin plate model underestimates the corresponding result that is derived from an analysis that takes into account shear deformations and effects of in-plane extension. Results will be presented corresponding to the case where \(\xi = 1\), representing the case where the shear tractions are identically zero at the interface. (Analogous results can also be developed for the case where the appropriate value of \(\xi\) is that applicable for the inextensible interface.)

Fig. 2 illustrates the variation in \(\Gamma_{FS}^R\) for specific combinations of \(\nu\) and \(\nu_f\), and for \(h/a = \eta \in (0.01, 0.5)\) and \((\mu_f / \mu)\) in \((1, 100)\). It is evident that, as \(\eta\) becomes larger and \((\mu_f / \mu) \rightarrow 1\), the estimate for the surface energy determined from the elementary thin plate model results in an appreciable error. It is also noted that the discrepancy between the two estimates becomes negligible when the shear modulus ratio \(\mu_f / \mu > 50\). This is due to the limiting values of expression (26) as either \(\eta \rightarrow 0\) or \((\mu_f / \mu) \rightarrow \infty\). Figs. 3 and 4 illustrate the corresponding results for the variation in \(\Gamma_{FL}^R\) for limiting combinations of \(\nu\) and \(\nu_f\), for \(h/a = \eta \in (0.01, 0.5)\), \((\mu_f / \mu) \in (1, 100)\) and three specific values of

\[
p_0 / \mu_f = 0.000005
\]

Fig. 4. Variation in the normalized surface energy for an elastic thin film–elastic substrate system with modular ratio \(\mu_f / \mu\), thickness ratio \(h/a\) and normalized stress \(p_0 / \mu_f\). Thin flexible film with effects of in-plane deformations \((\nu = \nu_f = 0.5)\).
Fig. 5. Variation in the normalized surface energy for an elastic thin film–elastic substrate system with modular ratio $\mu_\ell/\mu_t$, thickness ratio $h/a$ and normalized stress $p_0/\mu_t$. Thin flexible film with effects of in-plane deformations.

conditions at the edge of the delaminated zone is made complicated owing to the bonded contact between a flexible structural element and an elastic continuum. A computational treatment of the problem, based on either finite elements or boundary elements, will also require the use of special elements that can accommodate the exact compatibility conditions at the unbonded boundary. The approach adopted in the paper for the solution of the delamination problem is to impose a relaxed boundary condition that includes a regular $1/\sqrt{r}$ stress singularity at the edge of the delaminated zone along with kinematic and/or traction boundary conditions at the bonded regions. Using such an approach, an approximate analysis can be conducted to accommodate deformations of the elastic substrate and flexure of the thin elastic film, including the influence of shear deformations and in-plane extension of the thin film during flexure. The analysis yields a set of approximate relationships that can be used to estimate the adhesion energy of the interface. The numerical results presented in the paper illustrate the importance of the selection of the appropriate plate model for the flexure of the thin film for purposes of interpreting the adhesion energy of the interface. The choice of the model for interpreting the adhesion energy should be dictated not only by the relative geometric characteristics of the thin film but also relative stiffness characteristics of the thin film to the substrate and the relative magnitude of the delamination pressure, in situations where the deflections of the thin film are sufficiently large to induce in-plane extension.

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Appendix A

This Appendix presents an approximate solution to the problem of the large deflection of a uniformly loaded thin plate that exhibits Hookean elastic behaviour. The solution is based on a variational procedure that takes into account the contributions from the flexural energy of the thin plate as well as the in-plane extension accompanying large deflections. For axisymmetric large deflections of the thin film, the flexural deflections and the stretching deformations are approximated, respectively, by relationships

\[ w(r) = C_0 a \left( 1 - \left( \frac{r}{a} \right)^2 \right)^2; \quad u(r) = a \left[ \left( \frac{r}{a} \right)^2 - \frac{r}{a} \right] \left( C_1 + C_2 \frac{r}{a} \right) \]

where \( C_0, C_1 \) and \( C_2 \) are arbitrary constants. The assumed forms of the deflection \( w(r) \) and the stretching of the mid-plane \( u(r) \) satisfy the condition of flexural fixity at the boundary of the plate \( r = a \) and the absence of radial displacements on both the axis of symmetry of the plate and at the boundary \( r = a \). The strain energy in the thin film due to flexure and in-plane extension are given by

\[ U_B = \pi D \int_0^a \left[ \left( \frac{d^2 w}{dr^2} \right)^2 + \frac{1}{r^2} \left( \frac{dw}{dr} \right)^2 + \frac{2v}{r} \frac{dw}{dr} \frac{d^2 w}{dr^2} \right] r \, dr \]

and

\[ U_E = \frac{2 \pi \mu h}{(1 - v_l)} \int_0^a \left[ \left( \frac{du}{dr} + \frac{1}{2} \left( \frac{dw}{dr} \right)^2 \right)^2 + \left( \frac{u}{r} \right)^2 \right. \\
\left. + \frac{2v}{r} \left( \frac{du}{dr} \right) \left( \frac{1}{2} \left( \frac{dw}{dr} \right)^2 \right) \right] r \, dr \]

respectively, where \( D = \mu h^3/(6(1 - v_l)) \) is the flexural rigidity of the plate, \( \mu \), and \( v_l \) are, respectively, the linear elastic shear modulus and Poisson’s ratio of the thin plate, and \( h \) is the plate thickness. The arbitrary constants \( C_1 \) and \( C_2 \) are first determined in terms of \( C_0 \), using the variational constraint that the total potential energy \( U_B \) is a minimum for an equilibrium condition, i.e.,

\[ \frac{\partial U_B}{\partial C_0} = \frac{\partial U_E}{\partial C_1} = \frac{\partial U_E}{\partial C_2} = 0 \]  

\[ (A4) \]

The strain energy \( U_E \) consistent with the equilibrium configuration can then be added to the flexural energy \( U_B \) to determine the total strain energy \( U \) of the thin film that exhibits large deflections and in-plane stretching as follows:

\[ U = \frac{32 \pi D C_0^2}{3} \left[ 1 + \left( \frac{C_0 a}{h} \right)^2 \right] \]

where \( \chi = \{7505 + (4250 - 2791v_l)v_l\}/35280 \). The unknown constant \( C_0 \) is determined by making use of the principle of virtual displacements, i.e.,

\[ \frac{dU}{\delta C_0} = 2 \pi \int_0^a p_0 \delta C_0 a \left[ 1 - \left( \frac{r}{a} \right)^2 \right]^2 r \, dr \]

\[ (A6) \]

This gives the result

\[ C_0 = \frac{p_0 a^2}{64D} \left( \frac{1}{1 + 2\chi(C_0 a/h)^2} \right) \]

\[ (A7) \]

The result \( (A7) \) is a cubic in \( C_0 \), which can be solved to express the single real root of \( C_0 \) explicitly in terms of \( p_0 \). If \( \phi = 3p_0(1 - v_l)/2h_0 < 1 \), and \( \eta = h/a \), the positive real root of Eq. \( (A7) \) is

\[ C_0 = \frac{-\sqrt{36\phi^3} + \sqrt{6} \left( 9\sqrt{\phi} + \sqrt{6\eta^8 + 81\phi^2} \right)^{2/3}}{6\sqrt{\eta} \left( 9\sqrt{\phi} + \sqrt{6\eta^8 + 81\phi^2} \right)^{1/3}} \]

\[ (A8) \]

The elastic energy of the thin plate experiencing large deflections that cause in-plane extension can now be expressed in terms of \( p_0 \) by substituting this value of \( C_0 \) in Eq. \( (A5) \). As an approximation, however, the energy component in Eq. \( (A5) \) due to stretching is retained, but the value of \( C_0 \) is approximated by the dominant term on the right-hand side of Eqs. \( (A7) \) and \( (24) \), with the understanding that that terms of order \( 2\chi(C_0 a/h)^2 \) and higher can be neglected. This provides the following estimate for the total strain energy of the thin film undergoing large deflections involving both flexure and stretching:

\[ U_{LD_{TF}} = \frac{\pi \rho f^6}{384D \left( 1 + \chi \left( \frac{p_0}{64Dh} \right) \right)} \]

\[ (A9) \]

References


