Concentrated loading of a fibre-reinforced composite plate: Experimental and computational modeling of boundary fixity

H. Nikopour*, A.P.S. Selvadurai

Department of Civil Engineering and Applied Mechanics, McGill University, Montreal, QC H3A 2K6, Canada

Abstract

This paper examines the flexural behavior of a locally loaded Carbon Fibre-Reinforced Polymer (CFRP) composite rectangular plate with different edge support conditions. The transversely isotropic elasticity properties of the unidirectionally reinforced laminae were determined experimentally and verified through computational modeling. The assembly of the laminae is used to construct a model of the plate. The layered composite CFRP plate used in the experimental investigation consisted of 11 layers of a poly-ester matrix reinforced with carbon fibres. The bulk fibre volume fraction in the plate was approximately 61%. The experimental results for the deflected shape of the plate are compared with computational results that take into account large deflections of the plate within the small strain range. It was found that the Representative Area Element (RAE) method developed for estimating the elastic properties of unidirectional fibre reinforced plates provides reliable estimates for the finite element modeling of the composite plate at the macro-scale.

© 2013 Elsevier Ltd. All rights reserved.

1. Introduction

Fibre-reinforced plates are used quite extensively in various engineering applications, ranging from infrastructure engineering, aerospace engineering, automotive engineering, marine structures and wind energy production [1–4]. In many of these applications, the high strength to weight ratio requirements are satisfied by the fibre reinforced composite.

This paper examines the flexural mechanics problem for a Carbon Fibre Reinforced Polymer (CFRP) laminated rectangular plate that is constrained at the edges. In particular, the edge support conditions were varied to include either (i) complete fixity that can be achieved by clamping the plate between rigid surfaces or (ii) partial fixity that can be achieved by incorporating a deformable rubber-like layer at clamped locations. The elasticity of boundary supports is a research area that has received limited attention, but it is a critical aspect in advanced applications of fibre-reinforced composites, involving connections to metallic and non-metallic components [5,6]. The research also involves computational modeling of the experiments, which takes into consideration large deflection but small strain flexural behavior of the plate.

The accurate theoretical and computational modeling of the mechanical behavior of laminated plates requires the correct specification of their mechanical properties. The composite action of a laminated plate is largely governed by the mechanical properties of the individual layers and these properties are in turn governed by the fibre–matrix constituent properties at the micro-scale (i.e. fibre scale, where the fibre diameter could be approximately 5–10 μm in comparison to the thickness of a plate which could be several millimeters). The simplest theoretical relationships used to estimate the effective properties of a composite are the Voigt and Reuss bounds that assume either the existence of uniform strain (upper bound-Voigt) or uniform stress (lower bound – Reuss) in the individual phases [7]. Improvements to these bounds were provided by a number of investigators; the work of Hashin and Rosen [8] is based on energy principles and the studies by Hill [9,10] are based on the self-consistent scheme. References to recent developments can be found in [11–13]. In a companion study [14], the effective transversely isotropic properties of a unidirectionally fibre-reinforced CFRP composite were investigated using a micro-mechanical evaluation of the irregular fibre arrangements combined with computational simulations. It was shown that the effective transversely isotropic elastic properties of the unidirectionally reinforced composite determined from experimental results together with computational simulations closely matched the results based on the theoretical relationships proposed by Hashin and Rosen [8].

This paper extends the research investigations to the modeling of rectangular laminated plates, consisting of eleven layers of unidirectionally reinforced elements with orthogonal fibre orientations. The plate can experience large deflection behavior during bending. The research also presents results of experimental investigations of the flexural behavior of a multi-laminate composite plate with either fixed or partially fixed boundary conditions, and illustrates...
the need for incorporation of the large flexural deflections of the plate to examine experimental results. The requirement for considering the influence of large deflection behavior is demonstrated through a comparison of computational and experimental results at specified locations of the plate.

2. Plate lay-up and material properties identification

The fibre-reinforced plates were supplied by Aerospace Composite Products (ACP), California, USA. The tested plate measured 460.0 mm × 360.0 mm × 2.4 mm. The composite lay-up was identified using a scanning electron microscopy. The fibre area fraction in each layer was determined using the image processing software available in the MATLAB™ software. The longitudinal elastic properties were measured in the laboratory by conducting tensile tests according to the ASTM D3039 standard, while the transverse properties were estimated using a representative area element-based computational approach.

Scanning Electron Microscopy (SEM): The details of the SEM technique together with MATLAB™ procedure used to image the scans for development of computational representations of the transverse section are described in Selvadurai and Nikopour [14]. Fig. 1 shows the scanned results for the physical arrangement of fibres in the plate. The plate consisted of 11 orthogonally oriented layers with a lay-up of [(90°/0°)]₂, 90°, 0°, 90°, (90°/0°)] relative to the longitudinal direction of the plate. The fibre volume fraction was approximately 61%.

Longitudinal Elastic Properties: The longitudinal elastic properties of the CFRP specimens were determined from a series of tension tests (ASTM D3039) on fibre-reinforced specimens measuring 200.0 mm × 25.0 mm × 1.4 mm. Two strain gauges, oriented at 0° and 90° to the fibre direction, were installed at the center of the specimens to monitor the longitudinal (1) and transverse (2) strains. These values were then used to calculate one value of Poisson’s ratio, ν_{12}, applicable to the composite. Fig. 2 shows the instrumentation and loading grips used in the tension test. The experimentally determined longitudinal Young’s modulus for a single lamina, E₁₁, was estimated at (138.26 ± 5.26) GPa and Poisson’s ratio, ν_{12}, was estimated to be 0.23 ± 0.01. The properties of both the fibre and matrix materials used in the fabrication of the composite plate were provided by the manufacturer and these are listed in Table 1.

Transverse Elastic Properties: The transverse Young’s modulus, E₂₂, Poisson’s ratio, ν_{23}, and the plane strain shear modulus, G_{23}, were identified using a computational simulation of a two-dimensional plane strain Representative Area Element (RAE). Region B, as shown in Fig. 1, was selected as a representative area, and the finite element model of that area was developed using the ABAQUS™ software. The fibres and the matrix were modeled as isotropic linearly elastic materials with the elastic constants shown in Table 1. It was also assumed that there was perfect bonding between the fibres and matrix. The RAE was subjected to suitable homogeneous strains to estimate, through an energy equivalence, the effective transverse elasticity properties of the composite. Details of the RAE method used to determine each of the transverse elasticity properties (E₂₂, ν_{23}, and G_{23}) of the composite region are given by Selvadurai and Nikopour [14] and Nikopour [15]. Table 2 shows predictions for the transverse elastic constants using the Hashin and Rosen model [8], which does not take into account any irregularity in the fibre arrangement, and the RAE method [14] that considers irregular fibre arrangements.
3. Experimental localized loading of a plate

The test facility used in the research is shown in Fig. 3. It was an adaptation and modification of an arrangement used by Selvadurai and Yu [16]. The facility was modified to accommodate testing of fibre-reinforced plates with variable boundary conditions applicable to a rectangular region. The boundary conditions included either (i) complete fixity or (ii) partial fixity on a combination of edges. The effective dimensions of the plate measured from the boundary of the plate were 365.0 mm × 332.0 mm × 2.4 mm. The plate was subjected to a monotonically increasing quasi-static load...
applied through a rigid stainless steel spherical indenter with diameter of 15 mm, which was connected to an electro-mechanical actuator (Parker™) with a capacity of 3750 N. Fig. 4 shows a schematic view of the test setup and steel grips arrangement used in the experiments. Details of the steel grips and the fixed and partially fixed boundary conditions are shown in Fig. 5. Four potentiometers (ATech™) with an accuracy of ±0.01 mm were positioned in the test setup to monitor the deflection of the plate at salient locations. The applied load and the resulting displacements were recorded using a digital data acquisition system (Measurements COMPUTING™) connected to a computer that incorporated the TracerDAQ™ software. The fixed boundary condition on an edge was achieved by clamping the CFRP plate directly between two steel plates of thickness 25.4 mm, using four 4 mm screws along each edge. The partial fixity at a boundary was attained by incorporating two natural rubber strips with a thickness of 3 mm on the upper and lower surfaces of CFRP plates as shown in Table 3

<table>
<thead>
<tr>
<th>Elastic coefficient</th>
<th>$D_{1111}$</th>
<th>$D_{2222}$</th>
<th>$D_{1122}$</th>
<th>$D_{2233}$</th>
<th>$D_{1212}$</th>
<th>$D_{2323}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>RAE method (GPa)</td>
<td>143.77</td>
<td>13.35</td>
<td>4.33</td>
<td>3.97</td>
<td>55.20</td>
<td>4.85</td>
</tr>
<tr>
<td>Hashin and Rosen (GPa)</td>
<td>142.08</td>
<td>13.86</td>
<td>4.42</td>
<td>3.84</td>
<td>55.20</td>
<td>5.01</td>
</tr>
</tbody>
</table>

Fig. 7. Mesh configuration and experimental and computational results for load versus deflection at position of loading for the CFRP plate having three fixed edges.
in Fig. 5(b) to form the boundary condition corresponding to partial fixity. At the start of each experiment, the steel screws were tightened to achieve a uniform initial compression strain of 10% in each rubber strip. In order to identify the mechanical properties of the rubber, a compression test was performed following the ASTM D945-06 guidelines. In the small strain range up to 30%, rubber is considered to be a nearly linearly elastic material with a Poisson’s ratio, \( v_r \approx 0.5 \) and Young’s modulus, \( E_r = 1.96 \, \text{MPa} \). The rubber displayed no evidence of creep or other time-dependent phenomena and this behavior is consistent with experimental results obtained in other investigations (Selvadurai [17] gives a value of \( G_r = 0.8267 \, \text{MPa} \) where \( G_r \) is the shear modulus of the rubber; Selvadurai and Shi [18] give average of \( G_r = 0.738 \, \text{MPa} \)). The rubber displayed no evidence of creep or other time-dependent phenomena and this behavior is consistent with experimental results obtained in other investigations (Selvadurai [17] gives a value of \( G_r = 0.8267 \, \text{MPa} \) where \( G_r \) is the shear modulus of the rubber; Selvadurai and Shi [18] give average of \( G_r = 0.738 \, \text{MPa} \)).

Central and specific eccentric loading positions were selected for performing flexure testing of the plate (Fig. 6) and sixteen different sets of boundary conditions included 8 fixed boundary types and 8 partially fixed boundary types of arrangements. All tests were performed in a displacement control mode in a laboratory where the temperature was maintained at approximately 20 °C, with a nominal variation of ±1 °C. The maximum applied displacement at the point of application of the load, \( \Delta_{\text{max}} \), was limited to 5 mm. The specified displacements were applied in a quasi-static manner at a displacement rate of \( \Delta = 0.025 \, \text{mm/s} \) mainly to minimize any dynamic effects. Each test was performed 3 times in both loading and unloading sequences to establish repeatability of the non-linear load–deflection results.

4. Finite element formulation of large deflection flexure of laminated plates

It was observed that the load–displacement behavior at the point of application of the load displayed a non-linear response as the deflection increased. This characteristic response is considered to be an effect that results from large flexural deflections of the plate. The non-linear theory of plates that accounts for large deflection effects was developed by von Karman and is well documented in the texts by Timoshenko and Woinowsky-Krieger [19] and Reddy [20]. The fundamental difference between a classical plate theory for isotropic plates [19–21], and a plate theory for
isotropic plates that experience large deflections and their equivalents for composites, is the incorporation of the higher order terms in the rotations and the inclusion of in-plane strains. The theory of large deflections of laminated plates has also been investigated by Bathe and Bolourchi [22], Kam et al. [23], Akhras and Li [24,25], Akhras et al. [26], Tanaka et al. [27] and Selvadurai and Nikopour [28]. A brief presentation of the basic equations is given in Appendix A.

5. Computational modeling of plates

The equations governing large deflection analysis of plates is currently an available option in the general-purpose finite element code ABAQUS™. Stiffness coefficients in the plane of isotropy [29,30] were determined using a computational approach [14] and each layer was modeled as a homogenous orthotropic material. Perfect bonding was assumed to exist between composite-to-composite layers. Table 3 presents the elastic orthotropic coefficients calculated using the effective estimates given by Hashin and Rosen [8] and those derived computationally using the RAE approach [14]. Upper and lower surfaces of the steel grips were considered to be completely fixed and bonded to the composite plate. The electro-mechanical actuator was loaded until the prescribed plate deflection of 5 mm was attained. The load applied during an experiment was monitored using an Omega™ load cell which had a maximum load capacity of 375 kN and an accuracy of ±1%. In the case of plates with partial boundary fixity, the plate was clamped until the strain in the individual rubber strips (Fig. 5(b)) was approximately 10%. The computational modeling of the entire system including the test plate, the rubber strips and the steel grips was performed using a standard 9-node Lagrangian element available in the element library of ABAQUS™. A convergence study was conducted to determine the convergence of the method in relation to the mesh size. Mesh configuration for two types of boundary conditions are presented in Fig. 7(a) and Fig. 8(a). Since several variables are involved in the modeling, including (i) the plate boundary conditions (ii) point of application

<table>
<thead>
<tr>
<th>Boundary condition</th>
<th>Fully fixed edge</th>
<th>Partially fixed edge</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Experimental (N)</td>
<td>Hashin–Rosen (N)</td>
</tr>
<tr>
<td></td>
<td>73</td>
<td>76</td>
</tr>
<tr>
<td></td>
<td>131</td>
<td>138</td>
</tr>
<tr>
<td></td>
<td>298</td>
<td>311</td>
</tr>
<tr>
<td></td>
<td>435</td>
<td>451</td>
</tr>
<tr>
<td></td>
<td>720</td>
<td>742</td>
</tr>
<tr>
<td></td>
<td>568</td>
<td>609</td>
</tr>
<tr>
<td></td>
<td>740</td>
<td>763</td>
</tr>
<tr>
<td></td>
<td>828</td>
<td>865</td>
</tr>
</tbody>
</table>

Table 4
Load required to achieve a 5 mm deflection at the point of application of the central load.

<table>
<thead>
<tr>
<th>Boundary condition</th>
<th>Fully fixed edge</th>
<th>Partially fixed edge</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Experimental (N)</td>
<td>Hashin–Rosen (N)</td>
</tr>
<tr>
<td></td>
<td>29</td>
<td>31</td>
</tr>
<tr>
<td></td>
<td>54</td>
<td>57</td>
</tr>
<tr>
<td></td>
<td>103</td>
<td>110</td>
</tr>
<tr>
<td></td>
<td>641</td>
<td>673</td>
</tr>
<tr>
<td></td>
<td>1078</td>
<td>1143</td>
</tr>
<tr>
<td></td>
<td>675</td>
<td>712</td>
</tr>
<tr>
<td></td>
<td>1079</td>
<td>1157</td>
</tr>
<tr>
<td></td>
<td>1405</td>
<td>1481</td>
</tr>
</tbody>
</table>

Table 5
Load required to achieve a 5 mm deflection at the point of application of the eccentric load.
of the load (iii) the Hashin and Rosen estimates for the elastic constants and (iv) the RAE-based estimates of effective elastic properties, attention will be focused on the presentation of results that highlight the influence of boundary fixity on the load–displacement response.

6. Computational and experimental results

Table 4 summarizes the computational and experimental results for the loads required to attain a maximum deflection of 5 mm at the point of application of the load for a plate with fixed and partially fixed boundary conditions and subjected to central loading. Table 5 presents similar results for plates under non-symmetric loading. Comparison of experimental and computational results for the load–deflection behavior for the eight selected loading configurations with various combinations of boundary fixity, are presented in Figs. 7 and 8.

As expected, plate configurations with partial boundary fixity conditions required a lower maximum load to obtain a prescribed deflection when compared to plate configurations with fixed boundary conditions. There was negligible hysteresis during each loading–unloading cycle and energy dissipation did not change significantly when rubber supports were incorporated at the boundaries. Plates with higher stiffness and no rubber supports showed a greater nonlinear load–deflection trend compared to less stiff plates. This characteristic response is considered to be an effect that results from large deflections of the plate.

For predictions of the load required to attain the 5 mm deflection, the FEM-RAE method provided results slightly closer to the experimental values compared to the results derived from the FEM-Hashin–Rosen technique. The FEM-RAE method had a maximum error of 5.6% at peak deflection, whereas the FEM-Hashin–Rosen method had a maximum error of 7.3%, for the case of four partially fixed edge boundary conditions. Since the only variable in the modeling was the effective properties, it can be concluded that the RAE method [14] provides a more accurate estimate of composite elastic properties at the micro-scale compared to the idealized model proposed by Hashin and Rosen [8].

Fig. 9 shows deflection curves for plates where the loading is central and the edges have either a fixed or partially fixed condition. The maximum deflection at the free edges and the load required to attain a deflection of 5 mm at the center of the partially fixed boundary configuration were, respectively, 4.2% and 30.3% lower compared to the plate with the regular fixed boundary.

7. Conclusions

The mechanical behavior of a CFRP composite plate, with various fixed and partially fixed boundary conditions, and subjected to either localized central or eccentric transverse loading was examined experimentally and modeled computationally. It was found that the RAE method developed for estimating the elastic properties of uni-directional fibre reinforced plates provides reliable estimates for the finite element modeling of the composite plate at the macro-scale. The limitation of the RAE approach is the need to perform micro-mechanical assessment (SEM) of the fibre arrangements that would enable the estimation of the effective orthotropic elastic constants based on reliable fibre configurations. The RAE approach, however, accounts for geometric features of the fibre configuration, which is absent in the Hashin–Rosen type estimates for effective properties. The fact that both approaches give very similar results suggests that the theoretical estimates can be used with confidence in instances where SEM data are unavailable. The other important observation is that loading configurations with higher stiffness and fixed boundaries were more likely to exhibit a nonlinear trend in the applied load-transverse deflection. Installation of rubber strips had little effect on the energy dissipation of composite plates under quasi-static cyclic loading; however, it had a significant effect in reducing the maximum load in the CFRP plates.

Acknowledgments

The work described in the paper was supported by a NSERC Discovery Grant awarded to A.P.S. Selvadurai. The authors are grateful to research assistants, Brian Siciliani and Adrian Glowacki, for their assistance in the fabrication of the experimental setup. The composite plates used in this research were supplied by Aerospace Composite Products, CA, USA; the interest and support of Mr. Justin Sparr, Executive Vice President, of ACP is greatly acknowledged.

Appendix A

For finite element formulation of large deflections of the plate, the origin of the x- and y-coordinates used is taken as the center of the plate (Fig. 6). The displacement field in a thin laminated plate undergoing large deflections is assumed to be of the form:

\begin{align}
    u_x(x, y, z) &= u_0(x, y) + z\psi_x(x, y) \\
    u_y(x, y, z) &= v_0(x, y) + z\psi_y(x, y) \\
    u_z(x, y, z) &= w(x, y)
\end{align}

(A.1)
where $u_x$, $u_y$, $u_z$ are the deflections in the $x$, $y$, $z$ directions, respectively. $u_{0x}$, $u_{0y}$, $w$ are the associated mid-plane deflections, and $\psi_x$, and $\psi_y$ are the rotations due to shear. The strain-displacement relations in the von Karman plate theory can be expressed in the form [22,27,28]:

$$
\begin{align*}
\varepsilon_{xx} &= \frac{\partial u_x}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 + \frac{\partial}{\partial x} \left( \frac{\partial w}{\partial x} \right) + Z \kappa_x \varepsilon_o + ZK_y, \\
\varepsilon_{yy} &= \frac{\partial u_y}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 + \frac{\partial}{\partial y} \left( \frac{\partial w}{\partial y} \right) + Z \kappa_y \varepsilon_o + ZK_y, \\
\varepsilon_{xy} &= \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} + Z \left( \frac{\partial \psi_x}{\partial x} + \frac{\partial \psi_y}{\partial y} \right) = \varepsilon_o + ZK_1, \\
\gamma_{xy} &= \frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} + Z \left( \frac{\partial \psi_x}{\partial x} + \frac{\partial \psi_y}{\partial y} \right) = \gamma_o + ZK_3, \\
\gamma_{yz} &= \frac{\partial \psi_y}{\partial z} + \frac{\partial \psi_z}{\partial y} + Z \left( \frac{\partial \psi_y}{\partial y} + \frac{\partial \psi_z}{\partial z} \right) = \gamma_o + ZK_1, \\
\gamma_{zx} &= \frac{\partial \psi_z}{\partial x} + \frac{\partial \psi_x}{\partial z} + Z \left( \frac{\partial \psi_z}{\partial x} + \frac{\partial \psi_x}{\partial z} \right) = \gamma_o + ZK_1.
\end{align*}
$$

(A.2)

where $\varepsilon_o$ is the transverse shear strain, $\varepsilon_o(\alpha = 4, 5)$ are transverse shear strains, and $\kappa_i(i = x, y, z)$ are bending curvatures. The associated symmetric second Piola-Kirchhoff stress vector $\sigma$ is

$$
\sigma = [a_x, a_y, a_z, a_r, a_s]^T.
$$

(A.3)

The constitutive equations for the plate can be written as

$$
\begin{align*}
\{ N_i \} &= \left[ A_{ij} \right] \left[ B_{ij} \right] \{ e_i \} \quad (i, j = x, y, z) \\
\{ Q_2 \} &= \left[ \overline{A}_{44} \right] \left[ \overline{A}_{45} \right] \{ e_i \} \quad (i, j = x, y, z) \tag{A.4a}
\end{align*}
$$

and

$$
\begin{align*}
\{ Q_2 \} &= \left[ \overline{A}_{44} \right] \left[ \overline{A}_{45} \right] \{ e_i \} \quad (i, j = x, y, z). \tag{A.4b}
\end{align*}
$$

The components $A_{ij}$, $B_{ij}$, $D_{ij}(i, j = x, y, z)$ and $\overline{A}_{ij}(i, j = 4, 5)$ are the in-plane, bending coupling, bending or twisting, and thickness-shear stiffness coefficients. $N_i$, $M_{i}$, $Q_1$ and $Q_2$ are the stress resultants defined by

$$
\begin{align*}
\{ N_i \} &= \int_{h/2}^{h/2} (1, z) \sigma dz \\
\{ Q_1 \} &= \int_{h/2}^{h/2} (\sigma_x, \sigma_y) dz
\end{align*}
$$

and

$$
\begin{align*}
\{ A_{ij} \} &= \sum_{m=1}^{NL} \int_{z_m}^{z_{m+1}} Q_{ij}^{(m)}(1, z, z^2) dz, \quad (i, j = x, y, z) \\
\overline{A}_{ij} &= \sum_{m=1}^{NL} \int_{z_m}^{z_{m+1}} k_{x} k_{z} Q_{ij}^{(m)} dz, \quad (i, j = 4, 5; \alpha = 6, \beta = 6 - j) \tag{A.6}
\end{align*}
$$

where $z_m$ is the distance from the mid-plane to the lower surface of the $m$th layer, $NL$ is the total number of layers, $Q_{ij}$ are material constants, and $k_{x}$, $k_{z}$ are the shear correction coefficients which are set as [28] $k_1 = k_2 = \sqrt{2} / 3$.

The basis of the formulation of the governing equations of the plate is the principle of minimum total potential energy, in which the total potential energy $\pi$ of the system is expressed as the sum of strain energy, $U$, and the potential energy, $P$:

$$
\pi = U + P \tag{A.7}
$$

where

$$
\begin{align*}
U &= \frac{1}{2} \int_{V_p} \varepsilon^T \sigma dV \\
\pi &= \int_{\Omega_p} q(x, y) w(x, y) d\Omega
\end{align*}
$$

and

$$
P = - \int_{\Omega_p} q(x, y) w(x, y) d\Omega. \tag{A.9}
$$

where $V_p$ is the plate volume, $q(x, y)$ the distributed load intensity, and $\Omega_p$ the plate region.

Using the strain-deformation Eq. (A.2) and the constitutive Eq. (A.4), Eq. (A.8) can also be expressed as a function of mid-plane displacements. Performing the through thickness integration, the strain energy can be rewritten as [28]:

$$
U = \frac{1}{2} \int \left[ \varepsilon^{(1)} A d\varepsilon + 2\varepsilon^{(1)} B \varepsilon + \kappa^T 1 D \kappa + \gamma^T 1 \xi_1 \right] dy \tag{A.10}
$$

Considering the laminated composite plate discretized into $NE$ elements, the strain energy and potential energy of the plate can be expressed in the form:

$$
U = \sum_{i=1}^{NE} U_e
$$

and

$$
P = \sum_{i=1}^{NE} \left\{ \int_{\Omega_e} q(x, y) w(x, y) d\Omega \right\}
$$

where $Q_e$, $U_e$ are, respectively, the element area and the strain energy per element.

The mid-plane displacements and rotations $(u_0, v_0, w, \psi_x, \psi_y)$ within an element are given as a function of $5 \times n$ discrete nodal deflections and in matrix form they are:

$$
\mathbf{u} = \sum_{i=1}^{n} \left( \Phi_i \right) \nabla \mathbf{u}_i = \Phi \nabla \mathbf{u}_i
$$

where $n$ is the number of nodes of the element; $\Phi_i$ are the shape functions; $1$ is a $5 \times 5$ unit matrix; $\Phi$ is the shape function matrix; $\nabla \mathbf{u} = \{ \nabla u_1, \nabla v_1, \nabla w, \nabla \psi_x, \nabla \psi_y \}^T$; and the nodal displacements $\nabla \mathbf{u}_i$ at a node are:

$$
\nabla \mathbf{u}_i = \{ u_{0x}, v_{0y}, w, \psi_x, \psi_y \}^T, \quad i = 1, \ldots, n
$$

The first variation of Eq. (A.7) in terms of the nodal displacements can be expressed as:

$$
\delta \pi = \delta (U + P) = \sum_{i=1}^{NE} \left[ \delta \nabla \mathbf{u}_i \mathbf{F}_i \nabla \mathbf{u}_i \right] - \sum_{i=1}^{NE} [\delta \nabla \mathbf{F}_i] = 0 \tag{A.14}
$$

where $F_i$, $\nabla \mathbf{u}_i$ and, $P_i$ are the element internal and nodal force vectors, respectively.

References
