On the development of instabilities in an annulus and a shell composed of a poro-hyperelastic material

A. P. S. Selvadurai and A. P. Suvorov

Department of Civil Engineering and Applied Mechanics, McGill University, 817 Sherbrooke Street West, Montréal, Quebec, Canada H3A 0C3

The paper investigates the development of instability in an internally pressurized annulus of a poro-hyperelastic material. The theory of poro-hyperelasticity is proposed as an approach for modelling the mechanical behaviour of highly deformable elastic materials, the pore space of which is saturated with a fluid. The consideration of coupling between the mechanical response of the hyperelastic porous skeleton and the pore fluid is important when applying the developments to soft tissues encountered in biomechanical applications. The paper examines the development of an instability in a poro-hyperelastic annulus subjected to internal pressure. Using a computational approach, numerical solutions are obtained for the internal pressures that promote either short-term or long-term instability in a poro-hyperelastic annulus and a poro-hyperelastic shell. In addition, time-dependent effects of stability loss are examined. The analytical solutions are used to benchmark the accuracy of the computational approach.

1. Introduction

The development of instabilities in hyperelastic materials is well known as is demonstrated by the inflation of a children’s party balloon. There is a threshold pressure that needs to be attained to initiate an instability, after which incremental pressures can result in hyperelastic deformations, sadly ending in rupture. An early investigation of the development of instability and bursting of sheets of both natural rubber and synthetic rubber, rigidly clamped along a circular...
boundary, was investigated in a pioneering study by Treloar [1]. The interesting observation links the burst pressure to the hardness of the rubber at the instant of rupture. The studies by Adkins & Rivlin [2] and Rivlin & Saunders [3] (see also [4–8]) develop theoretical estimates for the inflation of membranes, similar to the experimental configuration of Treloar [1], in terms of a rational constitutive theory applicable to incompressible hyperelastic sheets with a strain energy function of the Mooney–Rivlin type. Inflation of spherical hyperelastic membranes is also obtained as limiting solutions in the studies performed by Green & Shield [9]. The work of Gent & Rivlin [10] describes the experiments conducted on hyperelastic tubes subjected to inflation, extension and twisting. The stability of a cylindrical membrane that is subjected to inflation and extension is also examined in a paper by Corneliussen & Shield [11] and further examined by Shield [12], Chen & Shield [13], Carroll [14]. Alexander [15] has also discussed experiments along similar lines. The works of Kydoniefs [16], Kydoniefs & Spencer [17, 18] and Li & Steigmann [19] deal with the inflation of cylindrical membranes and toroidal membranes with either a circular or a flattened cross section and a paper of related interest is due to Pipkin [20]. Roxburgh [21] presents the inflation of a nonlinearly deformed annular membrane. A variational approach for examining the toroidal membrane problem has also been presented by Tamadapu & Dasgupta [22]. The studies by Hill [23, 24], Abeyratne & Horgan [25] and Mangan & Destrade [26] investigate the development of instabilities in spherical shells. The works of Haughton & Ogden [27–30] and Haughton [31, 32] are systematic studies of the development of instabilities in cylindrical and spherical shells of hyperelastic materials. Fu & Xie [33] and Fu et al. [34] also discuss the inflation instability problems for cylindrical tubes. The development of instabilities in hyperelastic membranes subjected to fluid loadings is presented by Haughton [35]. Selvadurai & Shi [36] document experimental studies related to the fluid pressure loading of circular membranes fixed along a circular rigid boundary with observations of a nonlinear pressure-volume response even at moderately large strains. The list of articles cited in this brief survey is not meant to be a complete catalogue of developments in this area; the reader is referred to [34, 37–42] for additional references.

The topic of the development of instabilities in cylindrical annular regions has applications in biomedical engineering in connection with the development of enlargements and aneurysms in arteries due to high blood pressure. The classical analyses of such problems largely focus on the applications of the theory of hyperelasticity to examine the mechanics of arterial tissues. The developments in this area are quite extensive, and the presentation of a comprehensive survey is not feasible within the scope of this paper. The topic is discussed in extensive collections of articles and volumes and references to important earlier developments and current advances can be found in [43–52]. An excellent article by Eftaxiopoulos & Atkinson [53] presents a comprehensive study of the problem of angioplasty in the context of hyperelastic modelling of the artery, including the presence of atherosclerotic plaque.

This paper examines the development of instabilities in hyperelastic materials where the pore space is saturated with an incompressible fluid. The mathematical theory of poroelasticity was first formulated by Biot [54] to model the mechanics of water-saturated soils undergoing infinitesimal strains. The infinitesimal theory employs Hookean elastic behaviour of the porous skeleton and Darcy’s Law to describe the flow of water through the porous skeleton. The theory has been extensively applied in the context of geomechanics [55–67] and in the modelling of the mechanics of bone [68]. The classical theory of poroelasticity has also been formulated within the framework of the theory of mixtures and these developments are presented by several researchers including Green & Steel [69], Crochet & Naghdi [70], Mills [71, 72], Mills & Steel [73] and Rajagopal & Tao [74] and references to further articles in this area are given by Green & Naghdi [75], Atkin & Craine [76], Shi et al. [77], Rajagopal et al. [78], Dai et al. [79], Pence [80] and Selvadurai & Suvorov [81]. The classical theory of poroelasticity has also been extended to include thermal effects and these are documented in a recent volume by Selvadurai & Suvorov [82].

A formal theory of poroelasticity applicable to elastic media undergoing finite strains was proposed by Biot [83]. The applications of the theory of poro-hyperelasticity to the development of formal mathematical solutions of canonical problems in poroelasticity have been somewhat
limited. In the formulation of a theory of poro-hyperelasticity, several aspects need to be addressed. The skeletal hyperelastic behaviour of the porous solid is usually modelled via a classical theory of hyperelasticity applicable to the material of interest. There are a number of hyperelastic materials that have been developed in the literature and these include the neo-Hookean, Mooney–Rivlin, Ogden and Gent models; extensive discussions of these developments are given by Hart-Smith & Crisp [84], Green & Adkins [5], Spencer [85], Treloar [7], Ogden [8], Gent [86], Gent & Hua [87], Beatty [38,39], Selvadurai [88] and Selvadurai & Suvorov [81]. The fluid flow through the porous skeleton can, in general, be modelled by treating the poro-hyperelastic medium as a multi-scale porous medium (figure 1). In this case, when the flow is referred to the mesoscale, the flow can take place in the void channels, and can be described by viscosity-dominated processes such as Stokes’ flow. If the flow process is further refined to include the micro-scale, the isolated fluid inclusions can experience pressure increases and the fluid transport can be governed by Fickian-type diffusive processes (figure 1). In this study, however, attention is restricted to a single pore scale and a single process-driven fluid flow. In this case, the fluid flow through the porous skeleton is typically described by Darcy’s Law applicable to the porous space.

The permeability of the porous medium will experience change as hyperelastic deformations take place [89]. To account for permeability changes that accompany hyperelastic deformations, it is necessary to conduct extensive experimental investigations that can also account for the evolution of anisotropic permeability. Such extensions are best addressed through computational treatments. Finally, the partitioning of the total stress between the porous skeleton and the pore fluid also needs to take into consideration the compressibilities of the individual phases. In classical poroelasticity, for example, this partitioning can depend on the Biot coefficient, which considers the compressibility of the porous skeleton and the compressibility of the material composing the skeletal fabric. In instances where these compressibilities are significantly larger than the compressibility of the pore fluid saturating the pore space, the partitioning follows the classical Terzaghi relationship between the total stresses, the skeletal stresses and pore fluid pressure. For soft tissues exhibiting poro-hyperelastic behaviour, the compressibility considerations largely suggest the use of the Terzaghi effective stress concept. It is important to note that consideration of compressibilities of the individual phases, especially compressibility of the solid phase, significantly complicates the formulation of the poro-hyperelasticity problem but such formulations have been developed by Gajo & Denzer [90], Uzuoka & Borja [91] and Sun [92].

The earlier applications of poro-hyperelasticity largely focused on potential applications of the theory through computational simulations. These include the studies by Spilker & Simon [93], Simon [45], Simon et al. [94] and Ayyalasomayajula et al. [95]. Recently, contributions to the mathematical developments in the formulation and solutions of problems of the theory of
fluid-saturated hyperelastic materials are made by a number of authors including Murad et al. [96], Baek & Srinivasa [97], Gajo [98], Duda et al. [99], Baek & Pence [100], Wineman [101], Demirkoparan & Pence [102], Demirkoparan & Merodio [103] and Huyghe [104]. Recent papers by Selvadurai & Suvorov [81,105,106] present analytical approaches to the solution of certain canonical problems in poro-hyperelasticity relevant to one-dimensional compression, shear and inflation of annular regions that serve as benchmarks for the calibration of computational approaches.

The present paper examines the development of instabilities in poro-hyperelastic annular regions subjected to inflation. The present work was motivated in part by the computational results given by Merodio & Haughton [107] and Alhayani et al. [108] in which axisymmetric bifurcation into bulging was described from the viewpoint of construction of an adequate finite-element model. Time-dependent stability effects, which are a distinctive part of a rate-dependent material such as a poro-hyperelastic material, were incorporated to the indicated finite-element approach. In the limiting situations, where the instability is generated as \( t \to 0 \) and as \( t \to \infty \), our solutions correspond to the conventional results applicable to the appropriate hyperelasticity problem. The incorporation of pore fluid pressure diffusion through the porous skeleton introduces the time-dependent development of instabilities that makes our results of general interest. The paper also uses analytical developments to examine the accuracy of computational approaches to the modelling of the time-dependent development of instabilities in poro-hyperelastic materials.

2. Constitutive modelling

The details of the constitutive modelling required for the formulation of the poro-hyperelasticity problem are presented in Selvadurai & Suvorov [81,105,106]. The basic equations are summarized for completeness. The total Cauchy stress \( \sigma_{ij} \) is partitioned in terms of the Cauchy stress in the hyperelastic porous skeleton \( \sigma'_{ij} \) (effective stress) and the isotropic stress state in the saturating fluid \( p \) (fluid pressure) such that

\[
\sigma_{ij} = \sigma'_{ij} - Bp \delta_{ij},
\]

where \( \delta_{ij} \) is Kronecker’s delta and \( B \) is the Biot coefficient set equal to unity. The justification for the stress partitioning (2.1) rests on the relative compressibility of the hyperelastic skeleton with respect to the compressibility of the material composing the skeletal fabric. The effective or skeletal Cauchy stress \( \sigma'_{ij} \) can be written in terms of the deviatoric \( s'_{ij} \) and isotropic \( p' \delta_{ij} \) components in the form

\[
\sigma'_{ij} = s'_{ij} - p' \delta_{ij},
\]

where \( p' \) is the effective pressure.

We adopt the conventional formulation for problems in finite elasticity theory (e.g. [5,8,51,85,109–115]) and denote the Cartesian coordinates of a point in the undeformed configuration by \( X_i (i = 1, 2, 3) \) (with \( X_1 = X; \ X_2 = Y; \ X_3 = Z \)). In the deformed configuration, the coordinates of the same point are denoted by \( x_i (i = 1, 2, 3) \) (with \( x_1 = x; \ x_2 = y; \ x_3 = z \)). The deformation gradient tensor \( F \) is given by

\[
F = f_{ij} = \frac{\partial x_i}{\partial X_j}.
\]

The left Cauchy–Green strain tensor \( \tilde{B} \) is defined by

\[
\tilde{B} = \tilde{F}^T \tilde{F},
\]

where

\[
\tilde{F} = J^{-1/3} F; \quad J = \det F.
\]
The strain energy function $U$ for the isotropic hyperelastic material is assumed to be of the form

$$U = U(\tilde{I}_1, \tilde{I}_2, J),$$

(2.6)

where $\tilde{I}_1$ and $\tilde{I}_2$ are, respectively, the first and second invariants of the strain tensor $\tilde{B}$. The deviatoric component of effective stress $s'$ is defined as

$$s' = \frac{2}{J} \text{dev} \left\{ \left( \frac{\partial U}{\partial \tilde{I}_1} + \tilde{I}_1 \frac{\partial U}{\partial \tilde{I}_2} \right) \tilde{B} - \frac{\partial U}{\partial \tilde{I}_2} \tilde{B} : \tilde{B} \right\}. \quad (2.7)$$

The isotropic component of the effective stress is defined as

$$p' = -\frac{\partial U}{\partial J}. \quad (2.8)$$

The choice of a strain energy function for describing the hyperelastic behaviour of the porous skeleton largely depends on the scope of the particular application. Experiments show that hyperelastic materials, which experience moderately large strains, can be modelled using a Mooney–Rivlin form of a strain energy function [36,116–119] but such models are restrictive in their abilities to investigate the development of instabilities. The experimental investigations of Treloar [1], Rivlin & Saunders [3] and the re-evaluation of experimental data (e.g. [15,120] and others) point to the applicability of more general polynomial and Ogden-type strain energy functions to adequately model the development of instabilities. In this paper, we select the general form of the strain energy function $U$ proposed by Ogden [120] to model the hyperelastic skeletal response.

Only one term characterizing the deviatoric response and one term characterizing the volumetric response are included in the present strain energy function of the Ogden type:

$$U = \frac{2\mu}{\alpha^2} (\lambda_1^\alpha + \lambda_2^\alpha + \lambda_3^\alpha) J^{-\alpha/3} - 3) + \frac{1}{D_1} (J - 1)^2$$

$$= \frac{2\mu}{\alpha^2} (\lambda_1^{2\alpha/3} - \lambda_2^{\alpha/3} - \lambda_3^{\alpha/3} + \lambda_1^{-\alpha/3} \lambda_2^{\alpha/3} \lambda_3^{-\alpha/3} + \lambda_1^{-\alpha/3} \lambda_2^{-\alpha/3} \lambda_3^{\alpha/3} - 3) + \frac{1}{D_1} (\lambda_1 \lambda_2 \lambda_3 - 1)^2, \quad (2.9)$$

where $\mu$, $D_1$ and $\alpha$ are material parameters, and $\lambda_1$, $\lambda_2$ and $\lambda_3$ are the principal stretches, i.e. the eigenvalues of the deformation gradient tensor $F$, and the Jacobian $J = \lambda_1 \lambda_2 \lambda_3$. The constants $\mu$ and $D_1$ can be defined in terms of the linear elastic shear modulus ($G$) and the bulk modulus ($K$) of the porous skeleton as

$$\mu = G, \quad D_1 = \frac{2}{K}, \quad \nu = \frac{3K - 2G}{6K + 2G} = \frac{3 - \mu D_1}{6 + \mu D_1}, \quad (2.10)$$

where $\nu$ is Poisson’s ratio. Thus, the parameter $D_1$ can be considered as the measure of compressibility of the material. If $\alpha = 2$, the strain energy function for the neo-Hookean material can be recovered from (2.9). The form of the strain energy function (2.9) cannot be directly applied to materials with $\alpha = 0$, but a material with the limiting case $\alpha = 0$ can still exist.

In what follows, we concentrate on materials with positive values of the parameter $\alpha$, i.e.

$$\alpha > 0. \quad (2.11)$$

To complete the constitutive modelling, we need to consider fluid flow through a poro-hyperelastic material. Attention is restricted to the case where the entire pore space of the hyperelastic skeleton is saturated with a fluid and we assume that fluid flow takes place as a result of a gradient in the reduced Bernoulli potential. In the case of slow flows through the pore space, the velocity potential can be neglected in comparison to the other contributions and the datum potential can also be neglected provided that the potential is measured with reference to a fixed datum [121,122].
Considering the principle of mass conservation, we have
\[-\nabla \cdot \phi (v_f - v_s) = \nabla \cdot \frac{\partial u}{\partial t}, \tag{2.12}\]
where \(\phi\) is the porosity, \(v_f\) is the velocity of the fluid in the pore space and \(v_s\) is the velocity of the solid skeleton of the porous material, \(\nabla\) is the gradient operator referred to the coordinates of a particle of fluid in the deformed configuration, and \(u\) is the displacement vector defined as \(u = x - X\). Thus, \(v_s = \frac{\partial u}{\partial t}\). The derivation of equation (2.12) is presented in Selvadurai & Suvorov [81] and will not be repeated here.

The movement of fluids within the poro-hyperelastic material can depend on the scale at which the flow process is being modelled. We assume that, at the continuum scale, the flow of the fluid through the isotropic hyperelastic skeleton can be described by an isotropic form of Darcy’s Law as
\[\psi(\mathbf{v}_f - \mathbf{v}_s) = -\frac{k}{\eta} \nabla p, \tag{2.13}\]
where \(k\) is the permeability, which is assumed to be a constant and \(\eta\) is the dynamic fluid viscosity. We note that when porous hyperelastic media are subjected to large strains, the porosity will change with the strain and the permeability will evolve with the alteration in porosity. In this study, however, we do not attempt to introduce the strain dependency in the permeability and consequently, the permeability of the porous hyperelastic material is assumed to remain constant.

By combining (2.12) and (2.13), we can obtain the governing equation for the fluid pressure
\[\frac{k}{\eta} \nabla^2 p = \nabla \cdot \frac{\partial u}{\partial t}. \tag{2.14}\]
In (2.14) \(\nabla^2\) is Laplace’s operator referred to the coordinates in the deformed configuration.

3. Description of the radially symmetric problem

We consider the time-dependent response of a poro-hyperelastic material consisting of a porous fabric where the pores contain a fluid. The material of the solid phase and the fluid itself are assumed to be incompressible. Owing to the existence of pores, the whole material can, however, display the traits of a compressible hyperelastic material, once the fluid is allowed to flow in and out of the porous region (Treloar [7], §7.10, p. 154).

We examine a poro-hyperelastic annular region, which, in the undeformed configuration, occupies a cylindrical annular region (figure 2)
\[A \leq R \leq B, \tag{3.1}\]
referred to the cylindrical polar coordinate system \((R, \Phi, Z)\). Here attention is restricted to the radially symmetric plane strain deformation of this cylindrical shell described by
\[r = r(R, t); \quad \phi = \Phi; \quad z = Z. \tag{3.2}\]

As a result of this deformation, the inner and outer boundaries of the shell move to new positions \(r = a\) and \(r = b\), respectively, and the deformed shell occupies the volume (figure 2)
\[a \leq r \leq b. \tag{3.3}\]

The principal stretches associated with the radially symmetric deformation are
\[\lambda_1 = \frac{\partial r}{\partial R}; \quad \lambda_2 = \frac{r}{R}; \quad \lambda_3 = 1, \tag{3.4}\]
where the indices 1, 2 and 3 refer to the radial, circumferential and axial directions respectively, and plane strain deformations of the annular region are implied by (3.4).
If the radial displacement \( u(R, Z) \) is defined such that
\[
u(R, Z) = r(R, Z) - R, \tag{3.5}
\]
then the principal stretches in the radial and circumferential directions can be evaluated as
\[
\lambda_1 = 1 + \frac{\partial u}{\partial R}; \quad \lambda_2 = 1 + \frac{u}{R}. \tag{3.6}
\]

We assume that a total radial compressive stress of magnitude \( P_A > 0 \) is applied at the inner boundary of the shell and is caused, for example, by pressurization of the interior region of the annulus due to the injection of fluid with the same properties as the fluid saturating the annular domain (figure 2). The inner boundary is assumed to allow fluid flow into and out of the shell. Thus, the boundary condition at the inner boundary of the shell can be written as
\[
\sigma_{rr}(R, t) = -P_A H(t), \quad p(R, t) = p_A H(t), \quad R = A, \quad \forall \ t > 0, \tag{3.7}
\]
where \( p \) is the pore fluid pressure inside the shell, \( p_A \) is the pore fluid pressure at \( R = A \), and \( H(t) \) is the Heaviside step function of time. Owing to the free drainage condition for the inner boundary surface \( R = A \), the fluid pressure \( p_A \) must be equal to \( P_A \).

The outer boundary is assumed to be free of externally applied stress and impervious or undrained. The boundary conditions at the outer boundary can be written as
\[
\sigma_{rr}(R, t) = 0, \quad \frac{\partial p}{\partial R} = 0, \quad R = B \quad \forall \ t > 0. \tag{3.8}
\]

The second boundary condition of (3.8) is obtained from Darcy’s Law (2.13) because the relative fluid velocity is zero at the impervious boundary. (The undrained condition at the outer boundary is not a requirement for posing the problem but prescribed for the purposes of defining the boundary conditions.)

From the strain energy function, the total Cauchy stress components can be obtained as
\[
\begin{align*}
\sigma_{rr} &= \sigma'_{rr} - p = \frac{\lambda_1}{f} \frac{\partial U}{\partial \lambda_1} - p, \\
\sigma_{\theta\theta} &= \sigma'_{\theta\theta} - p = \frac{\lambda_2}{f} \frac{\partial U}{\partial \lambda_2} - p, \\
\sigma_{zz} &= \sigma'_{zz} - p = \frac{\lambda_3}{f} \frac{\partial U}{\partial \lambda_3} - p,
\end{align*}
\tag{3.9}
\]
and

**Figure 2.** Initial and deformed configurations of the porous fluid-saturated cylindrical shell. The pressure is applied at the inner boundary of the shell. (Online version in colour.)
where \( p \) is a pore fluid pressure. Using the expression for the strain energy function (2.9), we obtain from (3.9)

\[
\sigma_{rr} = \frac{2\mu}{3\alpha_1} \frac{\lambda_2}{\lambda_1} \left[ \frac{1}{\lambda_2 - \lambda_1} - 1 \right] \left( \frac{2\lambda_1 - \lambda_2}{2\lambda_1 - \lambda_2} - 1 \right) + \frac{2}{D_1} (\lambda_1 \lambda_2 - 1) - p, \\
\sigma_{\theta\theta} = \frac{2\mu}{3\alpha_1} \frac{\lambda_2}{\lambda_1} \left[ \frac{1}{\lambda_2 - \lambda_1} - 1 \right] \left( \frac{\lambda_1}{\lambda_2} - 2 \right) + \frac{2}{D_1} (\lambda_1 \lambda_2 - 1) - p, \\
\sigma_{zz} = \frac{2\mu}{3\alpha_1} \frac{\lambda_2}{\lambda_1} \left[ \frac{1}{\lambda_2 - \lambda_1} - 1 \right] \left( \frac{\lambda_1}{\lambda_2} - 2 \right) + \frac{2}{D_1} (\lambda_1 \lambda_2 - 1) - p.
\]

(3.10)

and

\[
\text{where the quantities with prime are the effective stresses.}
\]

Consequently, we evaluate

\[
\sigma_{rr} - \sigma_{\theta\theta} = \frac{2\mu}{\alpha} \left( \frac{2\alpha_1}{\lambda_2} - \frac{\lambda_2}{\lambda_1} \right),
\]

(3.12)

Using the chain rule for differentiation, we obtain

\[
\frac{\partial \sigma_{rr}}{\partial r} = \frac{\partial \sigma_{rr}}{\partial \lambda_1} \frac{\partial \lambda_1}{\partial r} + \frac{\partial \sigma_{rr}}{\partial \lambda_2} \frac{\partial \lambda_2}{\partial r} = \frac{\partial \sigma_{rr}}{\partial \lambda_1} \frac{\partial^2 \sigma_{rr}}{\partial r^2} \lambda_1 - 1 + \frac{\partial \sigma_{rr}}{\partial \lambda_2} \frac{1}{R} (1 - \lambda_1^{-1} \lambda_2),
\]

(3.13)

because

\[
\frac{\partial f}{\partial r} = \frac{\partial f}{\partial \lambda_1} \frac{\partial \lambda_1}{\partial R} = \frac{1}{R} (\lambda_1 - \lambda_2), \quad \frac{\partial \lambda_1}{\partial R} = \frac{\partial \lambda_2}{\partial R} = \frac{1}{R} (\lambda_1 - \lambda_2) \frac{\partial \lambda_1}{\partial \lambda_2},
\]

(3.14)

Thus, the single equation of equilibrium (3.11) can now be written as

\[
\frac{2}{D_1} \left( \frac{\partial^2 \sigma_{rr}}{\partial r^2} \lambda_2 + \frac{\lambda_1}{R} (\lambda_1 - \lambda_2) \right) + \frac{\partial^2 \sigma_{rr}}{\partial r^2} \left( \frac{2\alpha_1}{3\alpha_2} - \frac{\lambda_2}{\lambda_1} \right) \left( \frac{2\alpha_1}{3\alpha_2} - 1 \right) \frac{\lambda_1}{\alpha_1} + \left( \frac{\alpha_1}{\alpha_2} + 1 \right) \frac{\lambda_2}{\alpha_2} + \left( \frac{\alpha_1}{\alpha_2} + 1 \right) \frac{\lambda_2}{\alpha_2} - 2 \left( \frac{\alpha_3}{3} - 1 \right) \lambda_1 - \left( \frac{2\alpha_1}{3\alpha_2} - 1 \right) \lambda_2 + \left( \frac{\alpha_1}{\alpha_2} + 1 \right) \frac{\lambda_2}{\alpha_2} - \frac{\partial \sigma_{rr}}{\partial R} - \frac{\partial \sigma_{\theta\theta}}{\partial R} \left( \frac{\lambda_1 - \lambda_2}{\lambda_1 - \lambda_2} \right) = 0,
\]

(3.15)

which constitutes a second-order nonlinear partial differential equation for \( R(R) \). Using identities (3.14), the equilibrium equation can also be expressed in terms of the stretches only

\[
\frac{2}{D_1} \left( (\lambda_1 - \lambda_2) \frac{\partial \lambda_1}{\partial \lambda_2} + \lambda_1 (\lambda_1 - \lambda_2) \right) + (\lambda_1 - \lambda_2) \left( \frac{\partial \lambda_1}{\partial \lambda_2} + \frac{2\mu}{\alpha} \left( \frac{\lambda_2}{\lambda_1} - 1 \right) \frac{\lambda_1}{\alpha_1} + \left( \frac{\alpha_1}{\alpha_2} + 1 \right) \frac{\lambda_2}{\alpha_2} + \left( \frac{\alpha_1}{\alpha_2} + 1 \right) \frac{\lambda_2}{\alpha_2} - 2 \left( \frac{\alpha_3}{3} - 1 \right) \lambda_1 - \left( \frac{2\alpha_1}{3\alpha_2} - 1 \right) \lambda_2 + \left( \frac{\alpha_1}{\alpha_2} + 1 \right) \frac{\lambda_2}{\alpha_2} \right) = 0.
\]

(3.16)

It follows that \( \lambda_1 = \lambda_2 \) is a trivial solution of the differential equation (3.16) if the fluid pressure is uniform.
The traction boundary conditions (3.71) and (3.81) are given by

\[
\frac{2\mu}{3\alpha} \lambda_1^{-(\alpha/3)-1} \lambda_2^{-(\alpha/3)-1} (2\lambda_1^{\alpha} - \lambda_2^{\alpha} - 1) + \frac{2}{D_1} (\lambda_1\lambda_2 - 1) - p = -P_A; \quad \text{at } R = A
\]

\[
\frac{2\mu}{3\alpha} \lambda_1^{-(\alpha/3)-1} \lambda_2^{-(\alpha/3)-1} (2\lambda_1^{\alpha} - \lambda_2^{\alpha} - 1) + \frac{2}{D_1} (\lambda_1\lambda_2 - 1) - p = 0; \quad \text{at } R = B. \quad (3.17)
\]

The mass conservation principle (2.12) applied to the radially symmetric problem gives

\[
- \left( \frac{\partial}{\partial r} + \frac{1}{r} \right) \phi (v_t - v_s) = \left( \frac{\partial}{\partial r} + \frac{1}{r} \right) \frac{\partial u}{\partial t}, \quad (3.18)
\]

where \( u \) is the radial displacement, \( v_t \) is the velocity of the fluid in the pore space, \( v_s \) is the velocity of the solid skeleton. Here \([\left( \partial / \partial r \right) + (1/r)]\) is the divergence operator written in the plane polar coordinate system.

Now, using Darcy’s Law (2.13) and the mass conservation principle (3.18), we can eliminate the relative velocity term \( \phi (v_t - v_s) \) to obtain the governing equation for the fluid pressure as

\[
k \frac{\eta}{\lambda} \nabla^2 p = \left( \frac{\partial}{\partial r} + \frac{1}{r} \right) \frac{\partial u}{\partial t}, \quad (3.19)
\]

where \( k \) is the permeability and \( \eta \) is the dynamic fluid viscosity. In (3.19) \( \nabla^2 \) is Laplace’s operator referred to the coordinates in the deformed configuration, i.e.

\[
\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right). \quad (3.20)
\]

All derivatives in (3.19) can be found in terms of \( R \) instead of \( r \) using the relationships

\[
\frac{\partial}{\partial r} = \lambda_1^{-1} \frac{\partial}{\partial R}; \quad \frac{\partial^2}{\partial r^2} = \lambda_1^{-2} \frac{\partial^2}{\partial R^2} - \lambda_1^{-3} \frac{\partial^2}{\partial R} \frac{\partial}{\partial R}; \quad \frac{1}{r} \frac{\partial}{\partial r} = \lambda_1^{-1} \frac{1}{\lambda_1 \lambda_2} \frac{\partial}{\partial R}. \quad (3.21)
\]

Therefore, equation (3.19), governing the fluid flow, can be rewritten as

\[
k \frac{\eta}{\lambda} \left( \lambda_1^{-2} \left( \frac{\partial^2 p}{\partial R^2} - \lambda_1^{-1} \frac{\partial^2}{\partial R^2} \frac{\partial p}{\partial R} \right) + \lambda_1^{-1} \frac{1}{r} \frac{\partial p}{\partial R} \right) = \left( \lambda_1^{-1} \frac{\partial}{\partial R} + \frac{\partial}{\partial t} \right) \frac{\partial}{\partial R} \log(\lambda_1 \lambda_2), \quad (3.22)
\]

where \( \dot{r} = \partial r / \partial t = \partial u / \partial t \).

Since the inner boundary of the cylindrical shell allows fluid flow and the outer boundary is assumed to be impervious, the fluid flow boundary conditions are given by

\[
p(A, t) = P_A; \quad \left( \frac{\partial p(R, t)}{\partial R} \right)_{R = B} = 0. \quad (3.23)
\]

We note that the boundary condition \( p(R = A) = P_A \) cannot be satisfied initially, at \( t = 0 \), if the pressure \( P_A \) is applied suddenly as a step function. In this case, the instantaneous response of the fluid-saturated cylindrical shell will be undrained and we can determine the resulting fluid pressure by considering the no-flow boundary conditions at both surfaces \( R = A \) and \( R = B \).

4. Stability for the instantaneous response of a fluid-saturated shell

Two types of stability loss are possible for poro-hyperelastic shells. The first one is associated with limit-point instability when the shape of the cylinder is preserved and the deformation remains radially symmetric for all values of the applied pressure. The critical applied pressure associated with the limit-point instability does not depend on the cylinder length because there is no involvement of the axial dimension in the formulation. The second type of instability is due to bifurcation into bulging where the cylinder, initially straight, loses its shape at the onset of bifurcation by expanding radially at one end and contracting radially at the opposite end once the load has reached the critical value. In the latter case, the deformation pattern is axisymmetric and the length of the cylinder becomes an important parameter upon which the critical applied pressure depends.
In this section, we focus on the stability of the instantaneous response of a fluid-saturated poro-hyperelastic cylindrical shell subjected to a sudden pressurization described by the boundary conditions (3.7) and (3.8). In actual finite-element calculations, the step function loading must be replaced by a high-rate loading and then the instantaneous response can be observed at the end of the load increase phase. Alternatively, the same type of response can be observed in a fluid-saturated shell that is loaded relatively slowly but has a very low permeability.

The instantaneous response of this shell, composed of incompressible constituents, must also be locally incompressible, i.e. the volume of any local region taken from that shell must be preserved. Consequently, the instantaneous response of such a body must also be undrained. We note that when using finite-element analysis, the undrained response can be obtained by prescribing the no-flow boundary conditions at both surfaces \( R = A \) and \( R = B \). The fluid pressure boundary conditions in (3.7) and (3.8) cannot be satisfied in this case.

(a) Limit-point instability

Consider first the radially symmetric type of deformation because it allows us to derive useful analytical estimates for the critical applied pressure. Since the Jacobian \( J = \lambda_1 \lambda_2 \) is equal to unity \((\lambda_3 = 1 \text{ at any point of the domain})\),

\[
\lambda_1 = \frac{1}{\lambda_2}. \tag{4.1}
\]

It follows that the product \( 2(\lambda_1 \lambda_2 - 1)/D_1 \) in the stress representation (3.10) is equal to zero (note that we implied that the compressibility parameter \( D_1 \) is not necessarily small, i.e. the skeleton of the porous body is, in general, a compressible fabric). Therefore, the stresses (3.10) can now be represented as

\[
\begin{align*}
\sigma_{rr} &= \frac{2\mu}{3\alpha} (2\lambda_2^{-\alpha} - \lambda_2^{\alpha} - 1) - p, \\
\sigma_{\theta\theta} &= \frac{2\mu}{3\alpha} (-\lambda_2^{-\alpha} + 2\lambda_2^{\alpha} - 1) - p, \\
\sigma_{zz} &= \frac{2\mu}{3\alpha} (-\lambda_2^{-\alpha} - \lambda_2^{\alpha} + 2) - p.
\end{align*} \tag{4.2}
\]

It is convenient to use the following shorthand notation for the stretch \( \lambda_2 \)

\[
\lambda = \lambda_2.
\]

Since the Jacobian is equal to unity, the total volume of the cylinder is preserved and thus

\[
\frac{b^2 - a^2}{B^2 - A^2} = 1.
\]

Hence, we obtain obvious connections between the stretches at the boundaries of the cylinder

\[
\begin{align*}
\lambda_a &= \left(1 + \frac{B^2}{A^2} (\lambda_b^2 - 1) \right)^{1/2}, \\
\lambda_b &= \left(1 + \frac{A^2}{B^2} (\lambda_a^2 - 1) \right)^{1/2},
\end{align*} \tag{4.3}
\]

where \( \lambda_a \) is the stretch \( \lambda_2 \) at the inner surface \( R = A \) and \( \lambda_b \) is the stretch \( \lambda_2 \) at the outer surface \( R = B \).

Using the chain rule for differentiation, we obtain

\[
\frac{d}{dr} = \frac{d\lambda}{dR} \frac{dR}{dr} \frac{d}{d\lambda} = \frac{1}{R} (1 - \lambda^{-1}_1 \lambda_2) \frac{d}{d\lambda} = \frac{1}{R} (1 - \lambda_2^2) \frac{d}{d\lambda}. \tag{4.4}
\]

Therefore, changing the independent variable from \( r \) to \( \lambda \), the equilibrium equation (3.11) can be written as

\[
\lambda(1 - \lambda^2) \frac{d\sigma_{rr}}{d\lambda} + (\sigma_{rr} - \sigma_{\theta\theta}) = 0. \tag{4.5}
\]
Substituting the stress components (4.2) into the equilibrium equation (4.5) results in
\[ \frac{d\sigma_{rr}}{d\lambda} = -\frac{2\mu}{\alpha} \left( \frac{\lambda^{-\alpha} - \lambda^{\alpha}}{\lambda(1 - \lambda^2)} \right). \] (4.6)

It can be easily shown that for large values of \( \lambda \) the stress admits the following representation:
\[ \sigma_{rr} \approx -\frac{2\mu}{\alpha(a-2)}(\lambda^{a-2} - \lambda_b^{a-2}), \quad \lambda \to \infty. \]

Hence, it immediately follows that if \( a < 2 \), the shell becomes unstable for large stretches because the radial stress, and, therefore, the applied pressure, can only decrease with the increase in the stretch \( \lambda \). In addition, using (4.3) for large values of the stretch, we can find that the applied pressure is
\[ P_A \approx \frac{2\mu}{\alpha(a-2)} \lambda^{a-2} \left( \frac{B^2}{A^2} - 1 \right), \quad \lambda_b \to \infty. \]

Therefore, the shell remains stable for \( a > 2 \) because the applied pressure increases when the stretch grows. The case \( a = 2 \) requires special attention but it is shown in the electronic supplementary material, appendix A, that the shell is unstable for \( a = 2 \).

After finding the total radial stress, the fluid pressure \( p \) in the shell can be found from (4.21) as
\[ p = \frac{2\mu}{3\alpha} (2\lambda^{-\alpha} - \lambda^{\alpha} - 1) - \sigma_{rr}. \] (4.7)

Note that the fluid pressure is non-uniform across the thickness of the shell. Therefore, according to Darcy’s Law, the relative velocity of the fluid with respect to the solid skeleton is not equal to zero. However, in spite of the non-zero velocity, the fluid relative displacement is still equal to zero because the fluid is not able to move instantaneously, during the instantaneous application of the load. Thus, the fluid is not able to escape the shell, which is consistent with our assumption of local zero volume change.

Since we know the radial stress, from (4.5) we can approximate the hoop stress \( \sigma_{\theta\theta} \) for large values of the stretch as
\[ \sigma_{\theta\theta} \approx \frac{2\mu}{\alpha} \lambda^a, \quad \lambda \to \infty. \]

The effective hoop stress can now be found from
\[ \sigma'_{\theta\theta} \approx \frac{2\mu}{\alpha} \lambda^a + p = \frac{4\mu}{3\alpha} \lambda^{a}, \quad \lambda \to \infty, \]
and it is tensile.

We note that the value of the radial stress grows as \( O(\lambda^{a-2}) \) when the stretch increases while the fluid pressure and the hoop stress grow much faster, as \( O(\lambda^a) \). Also, note that neither the stresses nor the fluid pressure depends on the elastic constant \( D_1 \), which characterizes compressibility of the skeleton. Detailed solutions of the differential equation of equilibrium (4.6) for the materials with \( a = 1 \) and \( a = 2 \) are presented in electronic supplementary material, appendix A.

Figure 3 shows the dependence of the applied pressure at the inner surface \( P_A \) on the stretch at the outer surface \( R = B \) for a shell of moderate thickness with \( A/B = 2/3 \). The skeleton of the porous fluid-saturated shell is hyperelastic with an Ogden-type strain energy function. The response is shown for various values of the parameter \( a \). The analytical solution, obtained after integration of equation (4.6) as explained in electronic supplementary material, appendix A, is shown with small circles and the finite-element solution is shown with solid lines. To obtain the finite-element solution, the commercial finite-element program ABAQUS was used. A two-dimensional axisymmetric formulation was used with eight linear elements across the thickness of the shell. To obtain a convergent solution for large values of the stretch in shells with unstable behaviour (i.e. \( a = 1 \) and \( a = 2 \)), displacement boundary condition instead of stress boundary condition and the no-flow boundary condition were prescribed at the inner surface \( R = A \). Then, after obtaining the solution, the stress at the inner surface was evaluated by dividing the reaction
inflated poro-hyperelastic cylindrical shell, $A < R < B$:  
$A/B = 0.6667$

Figure 3. The applied pressure versus the stretch for an inflated fluid-saturated poro-hyperelastic cylindrical shell (instantaneous response). (Online version in colour.)

force at one of the intermediate nodes of the inner surface by the element size in the axial direction and by the length of the circumference of the inner surface, i.e.

$$P_A = \frac{RF_{\text{node}}}{2 \pi (A + u_{\text{node}}) h_{\text{elem}}},$$

where $RF_{\text{node}}$ is the reaction force at the node in the radial direction, $u_{\text{node}}$ is the displacement of the node and $h_{\text{elem}}$ is the element size in the axial direction. This indirect method of stress evaluation has a higher accuracy compared to the standard stress evaluation procedure using the values at Gauss points of the element because the reaction forces at the nodes are evaluated more accurately than the stresses at Gauss points of the elements. A good match between the finite-element and analytical solutions was obtained.

It can be observed from figure 3 that if $\alpha \leq 2$ the instantaneous response of poro-hyperelastic shell becomes unstable for a certain value of the applied pressure but when $\alpha > 2$ the behaviour is stable with respect to radially symmetric deformation as the applied pressure can grow unboundedly with the stretch.

Figure B1 is placed in electronic supplementary material, appendix B, and shows the fluid pressure $p$ at the inner $R = A$ and outer $R = B$ surfaces of the shell as a function of the stretch at $R = B$.

(b) Axisymmetric bifurcation

Next, we consider the instability associated with axisymmetric bifurcation (bulging). Motivation for studying this type of instability is the available analytical result for the stretch $\lambda_2$ at which an incompressible hyperelastic cylindrical shell experiences bifurcation [107,108]. Only the
membrane approximation for the value of the critical stretch $\lambda_2$ is available, i.e. the cylindrical shell is assumed thin. However, using this membrane approximation, it is possible to qualitatively explain unstable behaviour of other types of shells, e.g. thick and compressible. Behaviour of an incompressible hyperelastic shell is also relevant to the study of the instantaneous response of the poro-hyperelastic shell because in both cases the volume of the shell is preserved.

To reproduce these analytical results, let us introduce the new quantity derived from the strain energy function $U(\lambda_1, \lambda_2, \lambda_3)$:

$$\hat{U}(\lambda_2, \lambda_3) = U(\lambda_2^{-1} \lambda_3^{-1}, \lambda_2, \lambda_3) = \frac{2\mu}{\alpha^2} (\lambda_2^{-\alpha} \lambda_3^{-\alpha} + \lambda_2^\alpha + \lambda_3^\alpha - 3). \tag{4.8}$$

It is evident that this definition can only be applied to incompressible materials since $\lambda_1 \lambda_2 \lambda_3 = 1$.

Using the membrane approximation, the critical value of the stretch $\lambda_2$ can now be found by solving the following equation [29,52]:

$$\lambda_2^3 \hat{U}'_3 (\lambda_2^2 \hat{U}'_2 - \lambda_2 \hat{U}'_2) - (\lambda_2 \lambda_3 \hat{U}'_3^2 - \lambda_2 \hat{U}'_2)^2 + \left(\frac{2\pi R}{L}\right)^2 \lambda_2^2 \lambda_3 \hat{U}_3' \hat{U}_3'' = 0, \tag{4.9}$$

where we have used the notation $\hat{U}'_2 = \partial \hat{U}/\partial \lambda_2$, $\hat{U}'_3 = \partial \hat{U}/\partial \lambda_3$, and $L/R$ is the ratio of cylinder length to radius in the undeformed configuration. By using the expression for the strain energy $\hat{U}$ given by (4.8) in (4.9) and setting $\lambda_3 = 1$, we can derive a more explicit equation for finding $\lambda_2 = \lambda_0$ for arbitrary values of the parameter $\alpha$:

$$\alpha + 1 + (\alpha - 1)(\alpha + 2)\lambda_2^\alpha + \alpha(\alpha + 1)\lambda_2^2 + (\alpha - 1)(\alpha - 2)\lambda_2^3 - \lambda_2^4 + \left(\frac{2\pi R}{L}\right)^2 \lambda_2^2 ((\alpha - 1)\lambda_2^{2\alpha} + 2\lambda_2^\alpha - (\alpha + 1)) = 0. \tag{4.10}$$

The solution of this equation strongly depends on the ratio $L/R$. For completeness, we also present the equation for the stretch corresponding to the limit-point instability [29]:

$$\lambda_2 \hat{U}'_2 - \hat{U}_2' = 0, \tag{4.11}$$

which can be simplified to

$$\alpha + 2 + (\alpha - 2)\lambda_2^{2\alpha} = 0. \tag{4.12}$$

When $\alpha = 1$ and the cylinder is very long, $L/R \to \infty$, we have from (4.10) $2 + 2\lambda_2^2 - \lambda_2^4 = 0$ to find the stretch associated with bifurcation. This gives $\lambda_2 = \sqrt{1 + \sqrt{3}} \approx 1.6529$. On the other hand, for limit-point instability, from (4.12) $\lambda_2 = \sqrt{\frac{1}{3}} \approx 1.7321$. Thus, bifurcation is the dominant mode of stability loss for very long cylinders. When $\alpha \to 0$, it can be shown that equations (4.10) for very long cylinders and (4.12) have identical solutions for the critical stretch: $\lambda_2 = \sqrt{\alpha} \approx 1.649$.

The applied pressure can be found using the formula $P_A = (H/R)\hat{U}_2'/(\lambda_2 \lambda_3)$, where $H$ is the thickness of the membrane in the undeformed configuration. Thus, by finding the derivative of this expression with respect to $\lambda_2$ and equating it to zero, we can find the maximum value of the applied pressure as $P_{A,\text{max}} = (H/R)\hat{U}_2'/\lambda_3$.

Figure B2 placed in electronic supplementary material appendix B, shows the values of the critical stretch $\lambda_2$ corresponding to initiation of bifurcation, computed using the membrane approximation (4.10).

For thicker shells, the critical applied pressure associated with bifurcation can be obtained using the finite-element method. Let us describe how bifurcation analysis can be performed for a cylindrical shell using the finite-element program ABAQUS. By using predefined linear ‘analytical field’, the distribution of the applied pressure was slightly perturbed in the axial direction of the cylinder, $0 \leq z \leq L$, so that it is no longer uniform but linear. In particular, compared to the uniform distribution of the pressure for some nominal value $P_A$, the pressure was set equal to $P_A$ at the mid-height of the cylinder, increased, say, by 0.05%, at the upper end surface of the cylinder, and decreased by 0.05% at the lower end surface of the cylinder. In this case, $P_A(Z) = P_A(1 + 0.0005(Z/L - 1))$. We note that this linear distribution of the applied pressure will
promote bifurcation into \( n = 1 \) mode, i.e. the bifurcated shape will be described by the function \( \cos(n \pi z/L) \), \( n = 1, 0 \leq z \leq L \). Higher-order distributions of the applied pressure will promote other modes of bifurcations with larger values of \( n \), but these are not relevant because they give higher values for the critical applied pressure.

When the applied pressure reaches the critical value, the stretches at the lower and upper-end surfaces of the cylinder become different even though the applied pressure differs only slightly. This can be a sign of initiation of bifurcation. Usually (but not always) this results in a considerable slowdown in the program execution, which can also be an indicator of stability loss. Even if the program continues to run without slowdown, the preceding onset of bifurcation usually can be detected by visualization of the cylinder’s shape and detecting the moment when the cylinder’s shape is no longer straight. (Typical differences in values of the stretch at the lower and upper-end surfaces of the cylinder can be observed in figures 7 and 8.)

In figure 3, the critical applied pressure associated with bifurcation for the instantaneous response of the poro-hyperelastic shell is shown with isolated markers. The results are shown for the ratio of cylinder length to radius \( L/\bar{R} = 4 \) and 1 and the shell is assumed thick with \( A/B = 2/3 \). These results are obtained using the finite-element method as explained above. It is clear from the figure that for longer cylinders the critical applied pressure is smaller than for shorter cylinders. Bifurcation into bulging is the dominant mode of instability for large values of the parameter \( \alpha \), but for smaller \( \alpha \) (\( \alpha = 1 \), for example), limit-point instability is the prevailing mode of stability loss for the two cylinders considered. For a shorter cylinder with \( L/\bar{R} = 1 \), we also observe (figure 3) that smaller values of the critical applied pressure (and hence, smaller values of the parameter \( \alpha \)) correspond to larger values of the stretch \( \lambda_2 \). When \( \alpha \) approaches 2, both types of instabilities give almost identical values for the critical applied pressure, i.e. \( P_A = G \ln(B/A) \), and the stretch \( \lambda_2 \) associated with initiation of bifurcation becomes very large when the cylinder is short. All these results are consistent with the results obtained for thin cylindrical shells using membrane approximation.

5. Long-term stability of a compressible poro-hyperelastic shell

In this section, we study the long-term stability of the poro-hyperelastic cylindrical shell whose inner surface is subjected to an applied pressure of magnitude \( P_A \). The long-term response of such a shell can be obtained when the magnitude of the applied pressure grows very slowly. Alternatively, the pressure can be applied at a rapid rate until it reaches the maximum value, after which it remains unchanged for a long time. We would like to know whether the response of such a shell is stable in the long term. In the long term, the fluid pressure in the shell becomes equal to the applied pressure

\[
p = P_A, \tag{5.1}
\]

and therefore uniform. Unlike the instantaneous response, the long-term response of the hyperelastic shell depends on all elastic parameters, including the compressibility parameter \( D_1 \). Consequently, Poisson’s ratio \( \nu \) of the skeleton of the porous material becomes an important factor.

As before, we distinguish between two instability modes: (i) limit-point instability for which the cylindrical shape of the shell is retained and (ii) bifurcation into bulging, where the shell loses its shape by increasing its diameter at one end of the cylinder and decreasing the diameter at another end. In the first case, the deformation remains radially symmetric, but when the bifurcation occurs, an axisymmetric formulation of the problem is required.

For the radially symmetric deformation mode, the equilibrium equation (3.15) can be simplified by setting \( \partial p/\partial R = 0 \) because the fluid pressure is uniform. Also, the traction boundary conditions (3.17) can be simplified by setting the fluid pressure at both surfaces equal to \( P_A \), i.e. \( p_A = p_B = P_A \). The resulting equations are equivalent to those for a hyperelastic cylindrical shell, without fluid, subjected to the tensile stress \( P_A \) on the outer surface \( R = B \) and zero stress on the inner surface \( R = A \).
Figure 4. Dependence of the critical applied pressure on the compressibility of the cylindrical shell, \( A \leq R \leq B \), with \( A/B = 2/3 \) and \( L/R_{av} = 4 \); the parameter \( \alpha \) in Ogden’s strain energy function is equal to 3. The skeleton of the poro-hyperelastic shell is compressible and the critical pressure is determined for the long-term response. (Online version in colour.)

A finite difference solution of the resulting one-dimensional boundary-value problem can be obtained with the help of Matlab built-in function bvp4c and it has been described by Selvadurai & Suvorov [105]. Here we focus mostly on the results provided by the finite-element method because it is a more general formulation that can accommodate general axisymmetric modes of deformation.

Figure 4 shows the dependence of the critical (or maximum possible) applied pressure \( P_A \) on the compressibility parameter \( G/K = \mu D_1/2 \) of the shell. The ratio \( A/B = 2/3 \), as before, and the parameter \( \alpha \) in Ogden’s strain energy function is equal to 3. Firstly, we note that when \( \alpha = 3 \) the critical pressure corresponding to the limit-point instability of shell in the long-term can be evaluated according to

\[
P_{A_{\text{max}}} = \left( \frac{\mu^2}{3D_1} \right)^{1/3} \ln \frac{B^2}{A^2} = \left( \frac{G^2 K}{6} \right)^{1/3} \ln \frac{B^2}{A^2}.
\]  

(5.2)

This result is derived in electronic supplementary material, appendix C. The critical pressure evaluated from (5.2) is shown in figure 4 by the dash-dotted line. The critical pressure does not depend on the fluid pressure value, i.e. it is the same for poro-hyperelastic and hyperelastic shells. All other lines in figure 4 correspond to the stability loss associated with bifurcation. In this case, the ratio of cylinder length to the mid-radius, \( L/R_{av} = 2L/(A + B) \), becomes an important parameter and here \( L/R_{av} = 4 \). We clearly observe that the critical pressure associated with bifurcation is significantly lower than the pressure associated with limit-point instability. This is especially apparent for materials with low compressibility when \( G/K \to 0 \). The bifurcation critical load for the poro-hyperelastic shell is rate-dependent and represented here by two solid lines: the lowest line
corresponds to the loss of stability in the long term (long-term response) and the horizontal line gives the critical pressure for the instantaneous response. The bifurcation critical load strongly depends on the compressibility of the shell for the long-term response but does not depend on \( G/K \) if the instantaneous response is of interest. The response of the hyperelastic shell (without fluid), shown in figure 4 by the dashed line, is rate-independent and close to the long-term response of the poro-hyperelastic shell. It is noted that the Biot coefficient for the considered poro-hyperelastic shell is equal to unity, because the phase constituents are incompressible. On the other hand, it is not difficult to prove that the hyperelastic shell can effectively be treated as a poro-hyperelastic shell for which the Biot coefficient is equal to zero. Since the difference in the critical applied pressures for these two extreme cases is insignificant, we can conclude that for all intermediate values of the Biot coefficient, smaller than unity, the critical applied pressure will approximately be the same as that for the poro-hyperelastic shell with the Biot coefficient equal to one.

Figure B3 placed in electronic supplementary material appendix B, shows the dependence of the critical applied pressure \( P_A \) associated with bifurcation on compressibility \( G/K \) for a shorter cylindrical shell with \( L/R_{av} = 1; \alpha = 3 \) and \( A/B = 2/3 \) as in figure 4.

Figure B4 placed in electronic supplementary material appendix B, shows the critical applied pressure of the shell with \( \alpha = 4, L/R_{av} = 4 \) and \( A/B = 2/3 \).

Figure 5 shows the critical applied pressure corresponding to the long-term stability of the poro-hyperelastic shell whose skeleton is a compressible material with Poisson’s ratio \( v = 0.4 \) (thus, parameter \( G/K = 0.2143 \)). The critical value of the applied pressure is plotted with respect to the parameter \( \alpha \) of Ogden’s strain energy function. Geometrical parameters of the shell are the same as before: \( A/B = 2/3 \) and \( L/R_{av} \) is either 1 or 4. The dash-dotted line corresponds to the limit-point instability and this critical pressure does not depend on the length of the cylinder. Moreover, for \( \alpha > 3 \) the critical pressure associated with the limit-point instability becomes infinitely large, and thus, this instability mode cannot be realized. Two solid lines correspond to the bifurcation into bulging and it is obvious that the critical pressure associated with bulging is smaller than the critical pressure associated with limit-point instability for some range \( \alpha > \alpha^* \), where the value \( \alpha^* \) depends on the length of the cylinder. The critical pressure associated with bifurcation is smaller for longer cylinders and the parameter \( \alpha^* \) is also smaller for longer cylinders.

6. Time-dependent stability of poro-hyperelastic cylindrical shell

In this section, we examine the time-dependent stability of the poro-hyperelastic cylindrical shell subjected to internal pressurization. We assume that a compressive stress of magnitude \( P_A > 0 \) is suddenly applied at the inner boundary of the shell and it is caused by pressurization of the interior region of the shell through injection of fluid. The applied pressure \( P_A \) is assumed to remain unchanged after that. The boundary conditions for this problem are given by (3.7) and (3.8).

As before, we are interested in the stability of such a shell with respect to radially symmetric deformations (limit-point instability) and bifurcation into bulging. Since the time-dependent response is of interest here, we also determine how much time is required for the shell to become unstable; this time obviously depends on the permeability of the shell \( k \). Solution of the time-dependent differential equations (3.15) and (3.22), relevant to radially symmetric deformation and limit-point instability, can be obtained using adaptive finite difference solver bvp4c available in Matlab. Solution details are outlined in electronic supplementary material, appendix D. Solution of the more general axisymmetric problem is obtained using finite-element software Abaqus.

For a numerical example, we consider a short cylindrical shell, \( A \leq R \leq B \), with \( L/R_{av} = 1.2 \), and \( A/B = 2/3 \), as before. The skeleton of the poro-hyperelastic shell is compressible and Poisson’s ratio is equal to 0.2, i.e. \( G/K = 0.75 \). First, we take a relatively small value for the applied pressure: \( P_A/G = 0.272 \). As follows from figure 5, if the cylinder is short, then for small values of the
long-term stability of poro-HE shell

inflated poro-hyperelastic cylindrical shell, $A < R < B$:
$A/B = 0.6667$; compressibility of HE material:
$G/K = 0.2143, \nu = 0.4$

Figure 5. Dependence of the critical applied pressure for the cylindrical shell, $A \leq R \leq B$, $A/B = 2/3$, on the parameter $\alpha$ in Ogden's strain energy function. The skeleton of the poro-hyperelastic shell is compressible with Poisson's ratio $\nu = 0.4$ and the critical pressure is determined for the long-term response. (Online version in colour.)

parameter $\alpha$ and for sufficiently small loads, the only possible mode of stability loss is the limit-point instability. This fact is also demonstrated in figure 6. In figure 6, we plot the stretch $\lambda_2 = a/A$ at the inner surface of the cylinder versus time $t$ for shells with different values of the parameter $\alpha$. For small values of $\alpha$ (1, 1.5, 2), the shell becomes unstable at some time during application of the pressure $P_A$ and this stability loss manifests itself in unbounded growth of the stretch $\lambda_2$. The shape of the cylinder, however, is preserved, i.e. it is a limit-point instability. When the parameter $\alpha$ increases (2.2, 2.5, 3), the behaviour of the shell becomes stable, i.e. the stretch goes to a well-defined limit as time increases. In fact, it can be calculated that for $\alpha = 2.2$, the critical applied pressure corresponding to the development of limit-point instability in the long-term is $P_A/G = 0.2741$, which is slightly larger than the applied pressure equal to 0.272.

The ratio of permeability to the square of inner radius, $k/A^2$, is taken equal to $7.5 \times 10^{-13}$. This is an important parameter because it controls, in particular, when the stability loss occurs and whether the shell loses its stability either closer to the instantaneous response or to the long-term response. Obviously, for this particular example, when $\alpha = 1$ the stability loss occurs almost instantaneously, but for $\alpha = 2$, the stability loss occurs when the shell approaches the long-term response.

Figure 7 shows the dependence of the stretch $\lambda_2$ on time when the applied pressure magnitude is $P_A/G = 0.36$. All other geometrical and physical parameters of the shell are the same as in figure 6. As follows from figure 5, when the load level is sufficiently high, an additional mode of stability loss arises, bifurcation into bulging, and this can be realized for some range of values of the parameter $\alpha$. This is also demonstrated in figure 7, where it is shown that the shell experiences bifurcation into bulging for the range of $\alpha$ roughly between 2 and 3.1. Bifurcation manifests itself in unequal values for the stretch $\lambda_2$ at the upper and lower end faces of the shell, i.e. sudden
increase in the stretch at the upper surface and decrease of the stretch at the lower surface or vice versa. In this way, the cylinder loses its original shape. As shown in figure 7, for the \( \alpha \leq 2 \), bifurcation is a time-dependent phenomenon and for larger values of the parameter \( \alpha \), bifurcation occurs at a later time.

The more detailed analysis shows that when \( P_A/G = 0.36 \) the shell remains stable in the long-term if \( \alpha \) is larger than 3.4; for example, when \( \alpha = 3.5 \) the critical applied pressure associated with bifurcation is 0.37, which is larger than the applied pressure value of 0.36. Thus, for \( P_A/G = 0.36 \), the shell will be stable if \( \alpha = 3.5 \).

It should be noted that the shell is unstable for \( \alpha \) smaller than 2. For these values of \( \alpha \), the shell loses stability instantaneously, when the load level has not yet reached the value \( P_A/G = 0.36 \). In fact, it can be calculated that the limit-point instability occurs instantaneously when the applied pressure is \( P_A/G = 0.309 \) for \( \alpha = 1 \), \( P_A/G = 0.331 \) for \( \alpha = 1.5 \) and \( P_A/G = 0.405 \) for \( \alpha = 2 \). Thus, it is obvious that when the applied pressure is \( P_A/G = 0.36 \), the shell will lose stability instantaneously for \( \alpha = 1 \) and 1.5. However, as shown in figure 7, the same type of response is expected for the shell with \( \alpha = 2 \), because upon application of the load \( P_A/G = 0.36 \), the shell almost instantaneously undergoes the deformation that looks large and unbounded. For \( \alpha = 2 \), the critical applied pressure associated with bifurcation will be slightly smaller than the pressure corresponding to limit-point instability, \( P_A/G = 0.405 \), and thus, strictly speaking, the shell will experience bifurcation ‘before’ the limit-point instability.

Figure 8 shows a three-dimensional image of half of the cylinder that was produced by sweeping the two-dimensional axisymmetric finite-element model along the circular arc by \( 180^\circ \). The initial and deformed configurations of the cylindrical shell are shown and the deformed configuration obviously gives the bifurcated shape with bulging. In the initial configuration, the
Figure 7. Dependence of the stretch $\lambda_2$ on time $t$ for the cylindrical shell, $A \leq R \leq B$, with $L/R_{av} = 1.2$ and $A/B = 2/3$. A constant pressure $P_A/G = 0.36$ is suddenly applied to the shell at $t = 0$. The skeleton of the poro-hyperelastic shell is compressible with Poisson's ratio $\nu = 0.2$.

Figure 8. Initial and deformed configurations of the cylindrical shell with $L/R_{av} = 4$ and $A/B = 2/3$. (Online version in colour.)
ratio of cylinder length to the average radius is taken equal to $L/R_{av} = 4$ and the ratio of the inner to outer radius of the cylinder $A/B = 2/3$.

7. Conclusion

In this paper, we studied the rate-dependent behaviour and stability of a poro-hyperelastic cylindrical shell subjected to sudden internal pressurization. The end surfaces of the cylinder were constrained against vertical motion but were free to move in the radial direction. The skeleton of the poro-hyperelastic shell is described by the Ogden’s type strain energy function that depends on three parameters: usual shear and bulk moduli, and in addition, the parameter $\alpha$. The fluid in the pore space of the shell as well as the material of the solid phase are assumed incompressible.

We showed that there are two modes of stability loss for the shell: the first one is limit-point instability, for which deformation remains radially symmetric for all possible values of the applied pressure, and the second one is axisymmetric bifurcation resulting in bulging over a length of the shell. In the latter case, when the instability occurs, the cylinder loses its shape by expanding in the radial direction at one end of the cylinder, and contracting at the opposite end.

We first studied the instantaneous response of the poro-hyperelastic shell subject to internal pressurization and determined the critical pressure for this response. We showed that the instantaneous response of the poro-hyperelastic shell is equivalent to the response of the incompressible hyperelastic shell, without the fluid, because the fluid pressure in the poro-hyperelastic shell is equal to the pressure (mean stress multiplied by $-1$) in the incompressible hyperelastic shell. Consequently, the stability of the instantaneous response of the poro-hyperelastic shell does not depend on the skeleton’s compressibility parameter (e.g. bulk modulus), included into the strain energy function. We demonstrated that the poro-hyperelastic shell may instantaneously exhibit the limit-point instability only when $\alpha < 2$ (figure 3). This result is in agreement with the well-known result that a necessary condition for the applied pressure to achieve a maximum for the incompressible cylindrical tube is that $|\alpha| < 2$ [14,29]. However, the poro-hyperelastic shell can experience (instantaneously) bifurcation into bulging for all values of $\alpha$. The critical applied pressure associated with bifurcation strongly depends on the ratio of cylinder length to average radius, $L/R_{av}$, and the critical pressure becomes smaller for longer cylinders. The critical pressure for bifurcation also depends on the parameter $\alpha$ and it decreases with the reduction of the parameter $\alpha$. The two modes of instability compete only for shorter cylinders; here, for small values of the parameter $\alpha$, approximately smaller than 2, the dominant mode of instability loss is the limit-point instability, but for larger values of $\alpha$, bifurcation is the only possible mode of instability. By contrast, for long cylinders, the only possible mode of instability for all $\alpha$ is bifurcation because it always occurs at stretch values lower than those for the radially symmetric mode. (The last conclusion are similar to those obtained by Haughton & Ogden [29] for regular hyperelastic materials, without the interstitial fluid, in that bifurcation can occur before or after the pressure maximum, depending on the length of the tube.)

We then examined the long-term response of the poro-hyperelastic cylindrical shell subjected to internal pressure of a fixed magnitude and determined the critical pressure for this response. Owing to the varying fluid content in a poro-hyperelastic material, the long-term response of the poro-hyperelastic shell must depend on the skeleton’s compressibility. The subject of stability of thick-walled compressible tubes for strain energy functions of general type is studied considerably less compared to the stability of thick-walled incompressible tubes [29,30], or compared to the stability of thick-walled compressible tubes composed of harmonic or Varga hyperelastic materials [25,32]. Bifurcation of compressible (spherical) membranes made of Ogden material studied by Haughton [31] is a useful simplification of the problem at hand but similar analytical results for thick-walled tubes have not been obtained. To this end, our research on the long-term response of the poro-hyperelastic shell of arbitrary thickness must be of interest. We first showed that the poro-hyperelastic shell may experience limit-point instability in the long-term only if $\alpha < 3$ (figure 5). Alternatively, one can say that a necessary condition for the applied pressure to achieve a maximum for a compressible cylindrical tube is that $\alpha < 3$ (restricting...
attention to positive $\alpha$). This result is new, to the authors’ knowledge. Then we focused mainly on bifurcation into bulging because the limit-point instability was already studied for $\alpha = 2$ by Selvadurai & Suvorov [105]. Our results clearly demonstrate that for a compressible poro-hyperelastic shell, the long-term critical pressure associated with bifurcation is lower than the critical pressure for the instantaneous response. A similar difference between critical pressures was obtained for the limit-point instability by Selvadurai & Suvorov [105]. As in the case of limit-point instability, the critical applied pressure associated with bifurcation noticeably depends on the compressibility of the skeleton, and it is smaller for materials that are more compressible. In the long-term, the response of the poro-hyperelastic shell, namely, the value of the critical applied pressure, is almost the same as for the regular hyperelastic shell: the critical pressure for the poro-hyperelastic shell is only slightly smaller than the corresponding value for the regular hyperelastic shell. The cases where $\alpha = 3$ and $\alpha = 4$ were studied in more detail. In particular, we showed that when $\alpha = 3$, the bifurcation into bulging is the dominant mode of instability, even for short cylinders with $L/R_{av} = 1$ (in which the critical pressure associated with bifurcation is expected to be larger, but it is still smaller than that associated with the limit-point instability). For $\alpha = 4$, bifurcation is the only possible mode of instability and there is no competition between the two instability modes. All other conclusions are the same as indicated at the end of the previous paragraph.

Finally, we examined the time-dependent response of the poro-hyperelastic cylindrical shell subjected to sudden pressurization at $t = 0$. The value of the applied pressure is fixed for $t > 0$. The cylinder was chosen short, with $L/R_{av} = 1.2$. When the applied pressure is below the critical pressure for the instantaneous response but higher than the critical pressure for the long-term response, the cylinder will not lose stability instantaneously, but will become unstable later, at some non-zero value of time (figures 6 and 7). The specific time for losing stability obviously depends on the magnitude of the applied pressure and on the permeability of the poro-hyperelastic skeleton. For smaller permeability, more time is required for the shell to lose stability. Here, we considered two instability modes as before. If the applied pressure is relatively small, for moderate and large values of the parameter $\alpha$ ($\alpha > 2$), the shell will be stable, and only for small values of the parameter $\alpha$, approximately smaller than 2, the limit-point instability will occur. When the pressure applied at $t = 0$ is increased, the lower bound on the values of the parameter $\alpha$ for which the shell remains stable shifts towards a higher value (say, $\alpha \geq 3.5$) because the additional mode of instability, associated with bulging, becomes possible for moderate values of the parameter $\alpha$, roughly between 2 and 3.5. The process of increasing the applied pressure can be continued and it can be shown that for any given value of the parameter $\alpha$ (maybe high), we can always find the sufficiently high applied pressure so that the shell experiences bulging at some time during load application. In all cases, for smaller and larger applied loads, if the instability occurs, it will occur later in time when the parameter $\alpha$ increases.

It is important to remark that the time-dependent stability loss will eventually take place only if the applied load is higher than the critical load corresponding to the long-term instability. In this sense, the critical applied pressure associated with the long-term response of the shell becomes a very important parameter.

Finally, we note that the finite-element method proved to be an effective tool for studying bifurcation into bulging for the poro-hyperelastic cylindrical shell. In contrast to a more sophisticated approach of Alhayani et al. [108], we did not introduce an imperfection in the geometry of the shell but used slightly perturbed distribution of the applied pressure along the length of the cylinder. Also, we did not use the Riks method for identifying the instability, as this method cannot be used for time-dependent poroelastic problems. Consequently, we could not concentrate on post-bifurcation behaviour of the shell, but to the best of our knowledge, the evolution of bifurcation due to radial expansion observed by Alhayani et al. [108] never occurred.

Data accessibility. This article has no additional data.
Authors' contributions. The development of the scope of the research, the formulation of the problems and the writing of the paper were completed by both authors (A.P.S.S. and A.P.S.). Computational and other aspects associated with the paper were completed by A.P.S. Both authors read and approved the manuscript.

Competing interests. The authors declare that they have no competing interests.

Funding. One of the authors (A.P.S.S.) thanks the Natural Sciences and Engineering Research Council of Canada, Discovery Grants Program and the James McGill Research Chairs Program of McGill University for the financial support.

References


