On the indentation of a poroelastic halfspace

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ABSTRACT

This paper re-examines the problem of the indentation of a halfspace region with Biot poroelastic properties by considering the displacement and pore fluid pressure boundary conditions that are consistent for adhesive contact at an impermeable interface. The mixed initial boundary value problems associated with the adhesive-impermeable indentation are reduced to a set of coupled Fredholm integral equations of the second-kind in the Laplace transform domain. These equations are solved to establish the time-dependent indentation response of the indenter. The results are supplemented by sets of bounds obtained by maintaining a zero radial displacement condition within the contact zone and either zero pore fluid pressure boundary conditions or impermeable conditions on the entire surface of the halfspace or impermeable conditions within the contact region but free-draining exterior to the indenter. The accuracy of the numerical schemes used in the solution of these poroelastic contact problems is also discussed.

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1. Introduction

The problem of the indentation of an isotropic elastic halfspace by a rigid circular indenter with a smooth flat base was first solved by Boussinesq (1885) by reducing the mixed boundary value problem in elasticity theory to a problem in potential theory. The resulting formulation yields an exact closed form result for the relationship between the force \( P \) acting on the indenter of radius \( a \) and the resulting rigid displacement \( \Delta \). i.e.

\[
P = \frac{4Ga\Delta}{(1 - \nu)}
\]

where \( G \) and \( \nu \) are the elastic constants. An alternative formulation of the Boussinesq indentation problem was developed by Harding and Sneddon (1945), where the contact problem is reduced to a system of dual integral equations that can also be reduced to a single integral equation of the Abel type through the application of finite Fourier transforms. This result is relevant to the current developments since the classical poroelastic contact problem should reduce to Boussinesq’s result when \( t \to 0 \) in which case \( \nu = 1/2 \); when \( t \to \infty \), the Poisson’s ratio corresponds to the skeletal value obtained for complete dissipation of pore fluid pressures. An alternative approach involves the use of the fundamental solution of Boussinesq (1885) for the action of a concentrated normal force on the surface of a halfspace (Selvadurai, 2000c, 2001) and the development of the integral equation for the unknown contact stress distribution within the contact zone. These approaches all yield the celebrated exact solution (i) for the indentation displacement (see, e.g., Aleynikov, 2010; Barber, 2018; Galin, 1961; Gladwell, 1980, 2008; Goodman, 2013; Green, 1949; Green and Zerna, 1968; Johnson, 1985;
Kachanov, Shafiro and Tsukrov, 2013; Ling, 1973; Lur’e, 1964; Selvadurai, 1979, 2004, 2007; Sneddon, 1965; Timoshenko and Goodier, 1970; Ullian, 1965 and others). A recent volume by Argatov and Mishuris (2018) also presents a very comprehensive and useful compilation of solutions to contact problems in classical elasticity and classical poroelasticity to examine the indentation of biological tissues. It should, however, be remarked that mechanics of biological tissues are better characterized by hyperelastic and fluid-saturated hyperelastic media that can undergo large elastic deformations (Green & Adkins, 1970; Green & Zerna, 1968; Ogden, 1986; Spencer, 1970; Taber, 2004; Truesdell & Noll, 1965). The extension to fluid saturated hyperelastic media was initiated by Biot (1972) and recent developments are given by Selvadurai and Suvorov (2016a, 2017, 2018) and Suvorov and Selvadurai (2016). The development of exact closed form analytical results for finite strain contact problems is highly unlikely. Certain analytical results can be developed for contact and inclusion problems in second-order elasticity theory applicable to elastic solids experiencing moderately large deformations (Green & Adkins, 1970; Green & Spratt, 1954; Rivlin, 1953; Selvadurai & Spencer, 1972; Signorini, 1942; Stoppelli, 1954). Solutions to certain classical contact and inclusion problems in second-order elasticity theory are given by Choi and Shield (1981), Guo and Kaloni (1994), Lindsay (1985, 1992), Sabin and Kaloni (1983), Selvadurai (1974, 1975), Selvadurai, Spencer and Rudyard (1988). There are in excess of 7000 publications on classical linear elastic contact problems and no attempt is made here to provide a comprehensive review of the topic. Only references germane to the specific topic investigated here will be discussed. The work on Boussinesq-type contact problems was extended by Lekhnitski (1963), Mossakovski (1954), Shield (1951), Svitlo (1970), Willis (1967) to include the effect of orthotropy and transverse isotropy. The topic of Boussinesq indentation of halfspace regions exhibiting axial variation in the elastic non-homogeneity was investigated by a number of researchers including Kassir and Chuapresert (1974), Koronev (1957), Mossakovski (1958), Selvadurai (1998b) and Selvadurai and Lan (1998), Szefer and Gaszynski (1975). This class of problem was also extensively studied in the context of functionally graded materials and layered elastic systems and references to these topics can be found in the relevant literature.

In all the above research on the classical axisymmetric problem of the indentation of a halfspace region by a circular indenter, the contact region is assumed to be frictionless. This renders the entire surface of the halfspace region free of shear tractions. The extension of the topic to include displacement constraints involves the specification of zero radial displacements in the contact zone. This changes the character of the elasticity solution that leads to oscillatory forms of the stress singularities within the contact zone. The axisymmetric contact problem of the adhesive contact between a rigid flat punch and an elastic halfspace was first developed by Mossakovski (1954) and Ullian (1956). Using a Wiener–Hopf/Hilbert transform technique (see e.g. Gladwell, 1980, 2008; Keer, 1975; Spence, 1968) for the solution of the mixed boundary value problem, these authors were able to evaluate the load-displacement relationship in the form

\[ P = 4Ga\Delta \left( \frac{\log(e^{(3-4\nu)} - 1)}{2-2\nu} \right) \]  

(ii)

In the limit when \( \nu \to 1/2 \), the fundamental solution for Boussinesq’s problem for the normal loading by a concentrated force is identical to Kelvin’s solution for the interior loading of an infinite space by a concentrated force, which, by virtue of the spatial symmetry of the problem, resembles a halfspace region with an inextensibility constraint. Hence as \( \nu \to 1/2 \), Eq. (i) equals Eq. (ii) and in the context of the poroelasticity problem, when the time \( t \to 0 \) the solution for the frictionless indentation problem is identical to the solution for the adhesive contact problem. This aspect was effectively used by Selvadurai (1989, 1993, 1994, 2000a, 2000b, 2003, 2009), Selvadurai and Au (1986) and Selvadurai, Singh and Au (1989) to (a) develop bounds for the displacement of inclusions embedded at the interface of bi-material regions and (b) examine the influence of the inextensibility constraint applied over the entire surface of the elastic halfspace region to develop bounds for the load–displacement behaviour of the indenter. Omitting details it can be shown that the axial load-displacement relation of the rigid circular indenter with a flat base can be bounded by the results

\[ \frac{1}{2(1-\nu)} \leq \frac{P}{8Ga\Delta} \leq \frac{2(1-\nu)}{(3-4\nu)} \]  

(iii)

When \( \nu = 1/2 \), the bounds converge to a single result. When \( \nu = 0 \), the bounds give

\[ 0.50 \leq \frac{P}{8Ga\Delta} \leq 0.67 \]

and the exact solution (ii) gives

\[ \frac{P}{8Ga\Delta} = \frac{\log 3}{2} \approx 0.549 \]  

(iv)

The above results indicate the limiting values for the load-displacement relationship for the poroelasticity problem when the contact conditions change from frictionless to adhesive. A further approach for the solution of the adhesive contact problem invokes the regular stress singularity at the boundary of the rigid indenter and reduces the problem to a single Fredholm integral equation of the second kind (Selvadurai, 1989). These studies clearly show that the oscillatory stress singularity resulting from the in-plane displacement constraint only marginally influences the estimation of the load-displacement relationship for the rigid indenter in adhesive contact with the halfspace. For the formulation of the adhesive contact problem that results in a Fredholm integral equation of the second-kind, the elastic axial load displacement relation for \( \nu = 0 \) is
given by
\[
\frac{p}{8G\alpha\Delta} \approx 0.546
\]

When \( \nu = 1/2 \), the elastic axial load displacement relation of the indenter obtained from the Wiener-Hopf/Hilbert transform approach is identical to that obtained via the reduction of the mixed boundary value problem to a Fredholm integral equation of the second-kind. The contact conditions described previously are limiting cases for situations where the in-plane boundary conditions are either traction prescribed, or displacement prescribed. In the context of the poroelasticity problem it is worth noting that both the skeletal stresses and the pore fluid pressures can exhibit oscillatory/regular forms of stress singularities when the mechanical boundary conditions change from displacement to traction (Atkinson & Craster, 1991; Craster & Atkinson, 1996) and the fluid pressure boundary condition changes from Dirichlet to Neumann on a plane boundary (Selvadurai, 2004, 2014).

A further type of boundary condition involves the consideration of frictional effects at the contact zone. This type of contact problem is most conveniently examined by employing computational approaches based on either finite element or boundary integral equation techniques. The analytical solutions for this class of contact problem are rare and approaches for examining the frictional contact problem were presented by Spence (1968, 1975) for the loading problem and by Turner (1979) for the unloading problem in the presence of Coulomb friction. An informative record of this topic is also given in Gladwell (2008). Recent contributions in this area are due to Zhupanska (2009) who examined the axisymmetric frictional contact between a rigid sphere and an elastic halfspace. Ballard and Jarušek (2011) examined the uniqueness issues associated with frictional contact problems exhibiting small friction, while Ballard (2013) provides an in-depth study of the application of variational inequalities to the study of the frictional contact problem.

Contact problems, where the constitutive behaviour of the halfspace region can be modelled by Biot’s theory of poroelasticity, have also been examined by a number of investigators. In the case of poroelastic behaviour, it is also necessary to consider the pore fluid pressure boundary conditions at the plane boundary of the halfspace in addition to the displacement and traction boundary conditions. A solution to the axisymmetric indentation of the surface of a poroelastic halfspace by a rigid circular indenter was presented by Heinrich and Desoyer (1961). The solution is not a rigorous development of the poroelastic contact problem since the normal contact stress on the indenter is represented by the expression for the normal stresses obtained for the Boussinesq indentation problem, which is only valid when \( t = 0 \) and \( t \to \infty \). The problem related to the indentation of a poroelastic halfspace by a spherical rigid indenter was examined by Agbezuge and Deresiewicz (1974); apart from the moving boundary nature of the contact region, these authors considered three classes of boundary conditions for the pore fluid pressure on the surface of the halfspace region. These included (a) a completely permeable surface, (b) a completely impermeable surface and (c) an indenter that is impermeable in the contact zone although the surface of the halfspace region is permeable. The basic methodology was then applied by Agbezuge and Deresiewicz (1975) to examine the poroelastic contact between a rigid cylindrical indenter with a flat base and a poroelastic halfspace. Chiarella and Booker (1975) also examined the axisymmetric indentation of a poroelastic halfspace with a porous surface by a frictionless rigid indenter. The set of dual integral equations obtained using a Hankel–Laplace transform approach is reduced to a double Volterra integral equation, which is solved via a Galerkin approach. The results of Agbezuge and Deresiewicz (1975) and Chiarella and Booker (1975) clearly indicate that the contact normal stresses are no longer constant with a Boussinesq form but vary with time. Gaszynski and Szefer (1978) presented a solution to the problem of the axisymmetric indentation of a poroelastic halfspace with pore fluid pressure boundary conditions that varied from null-Neumann to null Dirichlet, respectively, in the contact and exterior regions. The formulation of the problem follows the conventional Hankel-Laplace transform developments with a numerical solution of the ensuing equations. Yue and Selvadurai (1995a) also re-examined the axisymmetric contact problem considering variations in the types of pore fluid pressure boundary conditions and provided comparisons between their results and those developed by Chiarella and Booker (1975). Solutions for the axisymmetric poroelastic contact problem based on a discretization of the contact pressure distribution was also presented by Kim and Selvadurai (2016) and, again, comparisons are provided between the discretization approach and available analytical approaches. The indentation of a poroelastic layer was examined by Selvadurai and Yue (1994) where plausible pore fluid pressure boundary conditions are imposed on the surface and base region of the poroelastic layer. The methodology follows the Hankel-Laplace transform approach, with the numerical solution of the resulting Fredholm integral equations of the second-kind and the numerical inversion of Laplace transforms. Yue and Selvadurai (1994) examined the poroelastic contact problem where the indenter is eccentrically loaded without provision for loss of contact. The interaction of two circular rigid indenters on a poroelastic halfspace was examined by Lan and Selvadurai (1996); this approach involved Fourier series representations of the governing equations followed by the application of a Laplace–Hankel transformation technique to reduce the problem to an infinite series of Fredholm integral equations of the second kind. The pore pressure boundary conditions are restricted to either free-draining or impermeable conditions on the entire surface of the halfspace region. The results for the single indenter problem were then compared with the approximate solution based on the approach proposed by Heinrich and Desoyer (1961). Of related interest are the solutions to a wide range of problems related to a rigid inclusion embedded in a poroelastic infinite space developed by Yue and Selvadurai (1995b) where the conventional Fourier-Hankel-Laplace transform techniques are used to reduce the problems to the solution of sets of Fredholm integral equations of the second-kind. Further studies related to poroelastic contact and inclusion problems that rely on integral equation approaches can be found in the articles by Cheng (2015) and Selvadurai and Suvorov (2016b), Selvadurai
(1996a, 2007). Of related interest is the contact problem for an infinite beam of finite width and a poroelastic halfspace that was examined by Selvadurai and Shi (2015).

An important omission in the great majority of studies described previously is the boundary condition related to the radial displacement at the contact region. In most formulations, the frictionless condition involves the specification of zero pore fluid pressure and zero shear tractions in the contact zone. When impermeable conditions are prescribed within the contact zone, the pore pressure boundary condition within the contact zone corresponds to the null-Neumann condition but the shear tractions are also prescribed as zero. This appears to be a mathematically correct but physically inconsistent set of boundary conditions. The focus of this paper is therefore to specify zero in-plane or radial displacements within the contact zone, respecting the adhesive contact conditions applicable to a rigid indenter in adhesive contact with a poroelastic half-space. The formulation of the axisymmetric adhesive-indentation of a rigid punch with a poroelastic halfspace is developed through the application of Hankel and Laplace transform techniques. The pore fluid pressure boundary conditions employ either (a) completely pervious conditions or (b) completely impervious boundary conditions or (c) impervious boundary conditions within the contact zone and free drainage conditions exterior to the contact zone. The equations governing the poroelastic contact problem are reduced to systems of coupled Fredholm integral equations of the second kind. The results for the axial displacement of the rigid indenter are used to develop numerical results for the Degree of Consolidation of the rigid indenter, which indicates the relative consolidation settlement of the indenter over time.

2. Governing equations

We restrict attention to the class of problems that are formulated in relation to a state of axial symmetry referred to the cylindrical polar coordinate system $(r, z)$. In such cases, the constitutive relations describing isothermal poroelasticity in terms of the total stress tensor $\sigma$ and the pore fluid pressure $p$ are given by (see e.g. Yue and Selvadurai (1995b), and Cheng (2015))

$$\sigma(r, z, t) = \frac{2G\nu}{(1 - 2\nu)}\text{tr}e(r, z, t)I + 2G e(r, z, t) + p(r, z, t)I$$  \hspace{1cm} (1)

In (1), $e$ is the infinitesimal strain tensor; $I$ is the identity matrix; $G$ and $\nu$ are, respectively, the linear elastic shear modulus and the Poisson’s ratio of the solid skeleton. It should also be noted that (1) assumes the Terzaghi (1925) relationship for the effective stress. Improvements to this effective stress law are included in the theory of poroelasticity developed by Biot (1941), which introduces the Biot coefficient, and thus modifies the pore pressure term in (1). For elastic skeletal behaviour, the Biot coefficient depends on the compressibility of the porous skeleton and the compressibility of the solid material composing the porous skeleton (Selvadurai, 2019). The Biot coefficient for elasto-plastic behaviour of the porous skeleton can be different from the purely elastic case (Suvorov & Selvadurai, 2019).

The infinitesimal strains in the porous skeleton are related to the displacement vector $u$ through the relationship:

$$e = \frac{1}{2}(\nabla u + (\nabla u)^T),$$  \hspace{1cm} (2)

where $(\nabla)^T$ denotes the transpose and $\nabla$ is the gradient operator. In the absence of body forces, the quasi-static equations of equilibrium in the poroelastic medium can be written as

$$\nabla \cdot \sigma = 0.$$  \hspace{1cm} (3)

The equations governing quasi-static fluid flow are defined by Darcy’s law, and, for a hydraulically isotropic poroelastic medium, these take the form

$$v^f - v^i = -\frac{k}{\mu} \gamma_p \nabla p,$$  \hspace{1cm} (4)

where $v^f$ is the velocity of the fluid, $v^i$ is the velocity of the porous skeleton, $k$ is the coefficient of permeability, $\mu$ is the dynamic viscosity of the fluid, and $\gamma_p$ is its specific weight.

We make the assumption that the velocity of the porous skeleton is considerably smaller than that of the permeating fluid. Thus, the continuity equation associated with quasi-static fluid flow can be written as:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v^f) = 0.$$  \hspace{1cm} (5)

The equations governing classical poroelasticity are hence characterized by three independent material parameters: (a) the drained or skeletal Poisson’s ratio $\nu$, (b) the skeletal shear modulus $G$, and (c) the fluid transport parameter $k$. The constitutive parameters have to satisfy certain thermodynamic constraints to ensure positive definiteness of the strain energy potential; these constraints can be expressed in the form $G > 0; \ -1 \leq \nu \leq 1/2$. Alternative but equivalent descriptions are given in Cheng (2015) and further references can be found in Selvadurai (1996a, 2007).

3. Poroelasticity with axial symmetry

In view of the class of axisymmetric problems that will be examined in this paper, it is convenient to summarize the fully coupled governing equations of poroelasticity. The dependant variables of the problem are the pore fluid pressure
Fig. 1. A poroelastic semi-infinite region indented by a rigid circular disk.

\[ p(r, z, t), \] and the skeletal displacements \( u_r(r, z, t) \) and \( u_z(r, z, t) \) in the radial and axial directions, respectively, which satisfy the following set of linear partial differential equations

\[
\left( \nabla^2 - \frac{1}{r^2} \right) u_r - (2\eta - 1) \frac{\partial \Theta}{\partial r} = \frac{1}{G} \frac{\partial p}{\partial r}, \tag{6}
\]

\[
\nabla^2 u_z - (2\eta - 1) \frac{\partial \Theta}{\partial z} = \frac{1}{G} \frac{\partial p}{\partial z}, \tag{7}
\]

\[
\frac{\partial \Theta}{\partial t} = c \nabla^2 \Theta, \tag{8}
\]

where

\[
\eta = \frac{(1 - \nu)}{(1 - 2\nu)}; \quad c = \frac{2G\eta k}{\gamma_w} \tag{9}
\]

In these equations, \( \Theta \) is the volumetric strain, and \( \nabla^2 \) is the axisymmetric form of Laplace’s operator given by

\[
\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \tag{10}
\]

3.1. The formulation of the initial mixed boundary value problems

We now examine problems pertaining to the time-dependant indentation of a rigid circular disk of radius \( a \) with a flat base that is bonded to a poroelastic semi-infinite medium (Fig. 1). The circular disk is assumed to be subjected to an axial load \( P_0 H(t) \), where \( H(t) \) is the Heaviside step function. The physical domain of the contact problem refers to a semi-infinite region, which is bounded within the region \( r \leq a; \ t > 0 \) and \( z \leq 0; \ t > 0 \). We consider the category of problems where the rigid indenter is in bonded contact with the poroelastic halfspace, thereby prescribing time-dependant axial displacement within the contact zone. The bonded contact ensures that the in-plane or radial displacement is zero within the contact zone. On the surface region exterior to the rigid indenter the surface of the halfspace region is unstressed.

Hence, the mixed boundary conditions applicable to the surface within the indentation region at \( z = 0 \) can then be written as:

\[
u_2(r, 0, t) = \Delta(t), \quad 0 \leq r \leq a; \quad t > 0 \tag{11}
\]

\[
\sigma_{zz}(r, 0, t) = 0, \quad a \leq r < \infty; \quad t > 0 \tag{12}
\]

\[
u_r(r, 0, t) = 0, \quad 0 \leq r \leq a; \quad t > 0 \tag{13}
\]

\[
\sigma_{rz}(r, 0, t) = 0, \quad a \leq r < \infty; \quad t > 0 \tag{14}
\]

The boundary condition (13) distinguishes this study from virtually all other treatments of the poroelastic contact problem, where the conventional assumption has been to prescribe zero shear tractions within the contact zone.
In order to complete the formulation of the boundary conditions necessary for the solution of the poroelastic contact problem, it is necessary to prescribe boundary conditions applicable to the pore fluid pressure on the surface of the halfspace region. These boundary conditions can be specified both within the contact zone and the region exterior to it; the choice of boundary conditions will be governed by the physical conditions relevant to the pore fluid pressures present on the contact zone and the region exterior to it. We further assume that the drainage conditions can be specified in relation to the pore fluid pressure and, in the specification of fluid velocities, the datum head component is neglected.

**Case I:** The simplest boundary condition assumes that the pore fluid pressures are zero over the entire surface of the halfspace. This would imply that the indenter is a porous disc even though it is bonded to the poroelastic halfspace. A further discussion of this type of boundary condition is presented in Selvadurai (2008). When the entire surface is completely permeable

\[ p(r, 0, t) = 0, \quad 0 \leq r < \infty; \quad t > 0 \]  

(15)

**Case II:** Here the boundary condition assumes that the entire surface of the halfspace is impervious; i.e.

\[ \frac{\partial}{\partial r} p(r, 0, t) = 0, \quad 0 \leq r < \infty; \quad t > 0 \]  

(16)

This assumption is an idealized situation where the surface of the halfspace contains an impermeable membrane preventing expulsion of pore fluids by hydraulic gradients.

**Case III:** In the third case, we assume that the interface between the bonded rigid indenter and the poroelastic region is impervious, but the region exterior to the contact zone is completely pervious; i.e.

\[ p(r, 0, t) = 0, \quad 0 \leq r \leq a; \quad t > 0 \]  

(17)

\[ \frac{\partial}{\partial z} p(r, 0, t) = 0, \quad 0 \leq r < a; \quad t > 0 \]  

(18)

This set of pore fluid pressure boundary conditions are consistent with the boundary conditions applicable to a bonded indenter.

In addition to these boundary conditions, the formulation of the poroelastic contact problem should also satisfy the following initial conditions

\[ \{\sigma(x, 0), u(x, 0), p(x, 0)\} = 0. \]  

(19)

For mathematical consistency relevant to the development of halfspace problems, the solutions should also satisfy the following regularity conditions

\[ \lim_{|x| \to \infty} \{\sigma(x, t), u(x, t), p(x, t)\} = 0. \]  

(20)

### 3.2. Solution of the governing equations

Solution of the partial differential Eqs. (6)–(8) can be approached in a variety of ways and these have been documented in the literature (see e.g. Kim and Selvadurai, 2016; Selvadurai, 2007; Yue and Selvadurai, 1995b). We adopt the approach proposed by McNamee and Gibson (1960a, 1960b) where the solution to the governing coupled partial differential Eqs. (6)–(8) can be represented in terms of two scalar functions \( S(r, z, t) \) and \( E(r, z, t) \), which satisfy the following partial differential equations

\[ \nabla^2 S(r, z, t) = 0, \]  

(21)

\[ c \nabla^4 E(r, z, t) = \nabla_i \frac{\partial}{\partial t} E(r, z, t). \]  

(22)

The displacements, total stresses and pore fluid pressure can be uniquely represented in terms of \( S(r, z, t) \) and \( E(r, z, t) \) as follows:

\[ u_z = -\frac{\partial E}{\partial z} + z \frac{\partial S}{\partial z} - S; \quad u_r = -\frac{\partial E}{\partial r} + z \frac{\partial S}{\partial r}, \]  

(23)

\[ p = 2G \left( \frac{\partial S}{\partial z} - \eta \Theta \right); \quad \Theta = \nabla^2 E, \]  

(24)

\[ \sigma_{rr} = \frac{\partial^2 E}{\partial r^2} - \nabla_i \frac{\partial^2 S}{\partial r \partial z} + \frac{\partial S}{\partial r}, \]  

(25)

\[ \sigma_{zz} = \frac{\partial^2 E}{\partial z^2} - \nabla_i \frac{\partial^2 S}{\partial z \partial r} + \frac{\partial S}{\partial z}; \quad \sigma_{rz} = \frac{\partial^2 E}{\partial r \partial z} - z \frac{\partial^2 S}{\partial r \partial z}. \]  

(26)
The representations (23)–(26) in terms of $S(r, z, t)$ and $E(r, z, t)$ can be verified by back-substitution into Eqs. (6)–(8). Analytical solutions for the time-dependant axisymmetric poroelasticity problem can be obtained using integral transformation techniques. Laplace and Hankel transforms are used to remove, respectively, time-dependency and dependency on the radial co-ordinate: i.e.

$$\tilde{F}(\xi, z, t) = \int_0^\infty r J_0(\xi r) F(r, z, t) dr,$$

$$\tilde{F}(r, z, s) = \frac{1}{2\pi i} \int_0^\infty e^{-st} F(r, z, t) dt,$$

where (3) refers to the zero-th order Hankel transform and (33) refers to the Laplace transform of a specific function; $J_0$ is the Bessel function of the first kind of order zero; $\xi$ and $s$ are Hankel and Laplace transform parameters, respectively. Upon successive applications of Hankel and Laplace transforms, the governing PDEs (6)–(8) can be reduced to the following ODEs for the transformed variables $\tilde{S}(\xi, z, s)$ and $\tilde{E}(\xi, z, s)$

$$\left(\frac{d^2}{dz^2} - \xi^2\right) \tilde{S}(\xi, z, s) = 0,$$

$$\left(\frac{d^2}{dz^2} - \xi^2\right) \left(\frac{d^2}{dz^2} - \left[\xi^2 + \frac{s}{c}\right]\right) \tilde{E}(\xi, z, s) = 0.$$

In the Hankel–Laplace space, the representation for the governing differential Eqs. (29) and (30) for the poroelastic half-space can be written as

$$\tilde{S}(\xi, z, s) = A_1 e^{-\xi z},$$

$$\tilde{E}(\xi, z, s) = A_2 e^{-\xi z} + A_3 e^{-wz},$$

where $A_1$, $A_2$, and $A_3$ are arbitrary functions of $s$ and $\xi$. In the above, $\varphi$ is defined by the following expression

$$\varphi = \sqrt{\xi^2 + \frac{s}{c}}.$$

Using the expressions for displacements and stresses (23)–(26) in conjunction with (31) and (32), the boundary conditions (11)–(18) give the following sets of integral equations in the Laplace transformed space. For Cases I and II, they are given as

$$\int_0^\infty \left\{ \left(\tilde{\delta}_1^m + 1\right) \tilde{N}_1(\xi, s) - \frac{1}{2\eta} \left(\tilde{\delta}_2^m + 1\right) \tilde{N}_2(\xi, s) \right\} J_0(\xi r) d\xi = \delta; \quad r \leq a; \quad i = I, II$$

$$\int_0^\infty \tilde{N}_1(\xi, s) J_0(\xi r) d\xi = 0; \quad r > a;$$

$$\int_0^\infty \left\{ -\frac{1}{2\eta} \left(\tilde{\delta}_2^m + 1\right) \tilde{N}_1(\xi, s) + \left(\tilde{\delta}_3^m + 1\right) \tilde{N}_2(\xi, s) \right\} J_1(\xi r) d\xi = 0; \quad r \leq a; \quad i = I, II$$

$$\int_0^\infty \tilde{N}_2(\xi, s) J_1(\xi r) d\xi = 0; \quad r > a;$$

and for Case III are

$$\int_0^\infty \left\{ \left(\tilde{\delta}_1^m + 1\right) \tilde{N}_1(\xi, s) - \frac{1}{2\eta} \left(\tilde{\delta}_2^m + 1\right) \left[\tilde{N}_2(\xi, s) + \tilde{N}_3(\xi, s)\right] \right\} J_0(\xi r) d\xi = \delta; \quad r \leq a$$

$$\int_0^\infty \tilde{N}_1(\xi, s) J_0(\xi r) d\xi = 0; \quad r > a$$

$$\int_0^\infty \left\{ \left(\tilde{\delta}_1^m + 1\right) \tilde{N}_2(\xi, s) - \frac{1}{2\eta} \left(\tilde{\delta}_2^m + 1\right) \left[\tilde{N}_1(\xi, s) - \tilde{N}_3(\xi, s)\right] \right\} J_1(\xi r) d\xi = 0; \quad r \leq a$$

$$\int_0^\infty \tilde{N}_2(\xi, s) J_1(\xi r) d\xi = 0; \quad r > a$$

$$\int_0^\infty \xi^2 \left[\tilde{\delta}_3^m \left[\tilde{N}_2(\xi, s) - \tilde{N}_1(\xi, s)\right] + \left(\tilde{\delta}_3^m + 1\right) \tilde{N}_3(\xi, s) \right] J_0(\xi r) d\xi = 0; \quad r \leq a$$

$$\int_0^\infty \xi \tilde{N}_3(\xi, s) J_0(\xi r) d\xi = 0; \quad r > a$$

where

$$\delta = \left(\frac{2\eta - 1}{\eta}\right) G \Delta.$$

(44)
By introducing the auxiliary functions introduced by Sneddon and Lowengrub (1969)

\[ \tilde{N}_1(\xi, s) = \int_0^a \tilde{\phi}_1(\rho, s) \cos(\xi \rho) \, d\rho; \]

\[ \tilde{N}_2(\xi, s) = \int_0^a \tilde{\phi}_2(\rho, s) \sin(\xi \rho) \, d\rho. \]

\[ \tilde{N}_3(\xi, s) = \frac{1}{\xi} \int_0^a \tilde{\phi}_3(\rho, s) \sin(\xi \rho) \, d\rho; \]

the coupled systems of dual integral Eqs. (34)–(37) and (38)–(43) can be reduced further. Functions \( \tilde{\phi}_1(\rho, s) \), \( \tilde{\phi}_2(\rho, s) \), and \( \tilde{\phi}_3(\rho, s) \) should satisfy the following conditions

\[ \lim_{u \to 0} \{ \tilde{\phi}_1(u, s), \tilde{\phi}_2(u, s), \tilde{\phi}_3(u, s) \} = 0. \]

Upon substitution of (34.0)–(34.2) in (33.3)–(33.8), and using the following identities

\[ \frac{d}{d\rho} \int_0^\rho \frac{r J_0(\rho \xi)}{\sqrt{\rho^2 - r^2}} \, dr = \cos(\xi \rho), \]

\[ \int_0^\rho \frac{r J_1(\rho \xi)}{\sqrt{\rho^2 - r^2}} \, dr = \frac{\sin(\xi \rho)}{\xi}, \]

\[ \frac{d}{d\rho} \int_0^\rho \frac{r J_2(\rho \xi)}{\sqrt{\rho^2 - r^2}} \, dr = \rho \sin(\xi \rho), \]

the two sets of dual integral Eqs. (34)–(37) can be reduced to two simultaneous Fredholm integral equations of the second-kind in the Laplace transform domain for Cases I and II, given by

\[ \tilde{\phi}_1(\rho, s) + \int_0^a \tilde{\phi}_1(\rho, s) \tilde{\Omega}_{11}(r, \rho, s) \, d\rho + \int_0^a \tilde{\phi}_3(\rho, s) \tilde{\Omega}_{12}(r, \rho, s) \, d\rho = 2\pi \delta; \quad r \leq a; \quad i = 1, \ II \]

\[ \int_0^a \tilde{\phi}_2(\rho, s) \tilde{\Omega}_{21}(r, \rho, s) \, d\rho + \tilde{\phi}_2(\rho, s) + \int_0^a \tilde{\phi}_3(\rho, s) \tilde{\Omega}_{22}(r, \rho, s) \, d\rho = 0; \quad r \leq a; \quad i = 1, \ II \]

and three simultaneous Fredholm integral equations for Case III:

\[ \tilde{\phi}_1(\rho, s) + \int_0^a \tilde{\phi}_1(\rho, s) \tilde{\Omega}_{11}(r, \rho, s) \, d\rho + \int_0^a \tilde{\phi}_2(\rho, s) \tilde{\Omega}_{12}(r, \rho, s) \, d\rho + \int_0^a \tilde{\phi}_3(\rho, s) \tilde{\Omega}_{13}(r, \rho, s) \, d\rho = \frac{2\pi \delta}{\rho}; \quad r \leq a \]

\[ \int_0^a \tilde{\phi}_2(\rho, s) \tilde{\Omega}_{21}(r, \rho, s) \, d\rho + \tilde{\phi}_2(\rho, s) + \int_0^a \tilde{\phi}_3(\rho, s) \tilde{\Omega}_{22}(r, \rho, s) \, d\rho + \int_0^a \tilde{\phi}_3(\rho, s) \tilde{\Omega}_{23}(r, \rho, s) \, d\rho = 0; \quad r \leq a \]

\[ \int_0^a \tilde{\phi}_3(\rho, s) \tilde{\Omega}_{31}(r, \rho, s) \, d\rho + \int_0^a \tilde{\phi}_3(\rho, s) \tilde{\Omega}_{32}(r, \rho, s) \, d\rho + \tilde{\phi}_3(\rho, s) + \int_0^a \tilde{\phi}_3(\rho, s) \tilde{\Omega}_{33}(r, \rho, s) \, d\rho = 0; \quad r \leq a \]

where the kernel functions \( \tilde{\Omega}_{ij}(r, \rho, s) \), \( i = I, \ II, \ III; \quad j, k = 1, 2, 3 \) are given in Appendix A.

The Laplace transforms of the total axial load and the unknown contact stress, respectively, can be written in the transformed domain using the following relations

\[ \tilde{P}_2 = -2\pi s \int_0^a \tilde{\phi}_1(\rho, s) \, d\rho; \]

\[ \tilde{\sigma}_{zz}(r, 0, s) = \frac{\tilde{\phi}_1(a, s)}{(a^2 - r^2)^{1/2}} - \int_r^a \frac{d}{d\rho} \left( \tilde{\phi}_1(\rho, s) \right) \frac{d\rho}{\sqrt{\rho^2 - r^2}}. \]

As is evident, the formulation and the resulting set of Fredholm integral equations invokes the admissibility of the regular stress singularity in the treatment of the poroelasticity problem for the bonded rigid indenter. As noted in the introduction, when the focus is on the estimation of the load-displacement relationship for the bonded indenter, the contribution from the oscillatory form of the stress singularity can be neglected. If a modification is to be made to introduce the time-dependant form of the oscillatory stress singularity, this will entail the solution of an eigenvalue problem in the Laplace transform domain.

4. Solution of the systems of Fredholm integral equations

The governing Fredholm integral equations of the second kind given in (52)–(53) and (54)–(56) are not amenable to an exact closed-form solution. As such, recourse should be made to an appropriate numerical method for solving such equations. This involves solving the system of simultaneous Fredholm integral equations of the second-kind to generate the
unknown functions $\tilde{\phi}_1 (r, s)$, $\tilde{\phi}_2 (r, s)$, and $\tilde{\phi}_3 (r, s)$ in the Laplace transform domain. For numerical treatment of the integral Eqs. (52)–(53) and (54)–(56), the interval $(0, a)$ is divided into $N$ equally-spaced sub-intervals with their endpoints defined as $m_i = (i - 1)a/N; \quad (i = 1, 2, 3, \ldots, N + 1)$, and the collocation points defined as $n_i = (m_i - m_{i+1})/2; \quad (i = 1, 2, 3, \ldots, N)$. The Fredholm integral equations in (52)–(53) and (54)–(56) can then be converted into a system of algebraic equations as follows

$$\sum_{i,j=1}^{N} \left[ \mathbf{\tilde{H}}_{pq}^{ij} \right] \left\{ \mathbf{\tilde{P}}_p \right\} = \left\{ \mathbf{\tilde{B}}_p \right\}; \quad p, q = 1, 2, 3 \quad (59)$$

where the components of the matrices $\left[ \mathbf{\tilde{H}}_{pq}^{ij} \right]$, $\left\{ \mathbf{\tilde{P}}_p \right\}$, and $\left\{ \mathbf{\tilde{B}}_p \right\}$ are defined as

$$\left[ \mathbf{\tilde{H}}_{pq}^{ij} \right] = \begin{cases} \delta_{ij} + \frac{1}{N} \tilde{\Omega}_{pq}(n_i, n_j); & i = j = 1, 2, \ldots, N; & p = q \\ \frac{1}{N} \tilde{\Omega}_{pq}(n_i, n_j); & i = j = 1, 2, \ldots, N; & p \neq q \\ \frac{1}{N} \tilde{\Omega}_{pq}(n_i, n_j); & i \neq j = 1, 2, \ldots, N \end{cases} \quad (60)$$

$$\left\{ \mathbf{\tilde{P}}_p \right\} = \left\{ \tilde{\phi}_1 (n_i), \tilde{\phi}_2 (n_i), \tilde{\phi}_3 (n_i) \right\}^T; \quad i = 1, 2, \ldots, N \quad (61)$$

$$\left\{ \mathbf{\tilde{B}}_p \right\} = \begin{cases} 2\delta/\pi; & p = 1; \quad i = 1, 2, \ldots, N \\ 0; & p = 2, 3; \quad i = 1, 2, \ldots, N \end{cases} \quad (62)$$

in which $\delta_{ij}$ is the Kronecker’s delta function.

The time-dependant unknown functions $\tilde{\phi}_1 (r, t)$, $\tilde{\phi}_2 (r, t)$, and $\tilde{\phi}_3 (r, t)$ are then obtained using Schapery’s (1962) numerical Laplace inversion technique. Alternative techniques for inversion of Laplace transforms are documented by Cohen (2007) and Cheng, Sidauruk and Abousleiman (1994). The convergence of the numerical integration becomes computationally time consuming as the integrands of the infinite integrals rapidly become oscillatory for very small time factors ($t^* = ct/a^2 \leq 0.01$). The accuracy of the computations can be improved by increasing the number of segments ($N$) to 750 to obtain a highly stable and convergent solution for the Fredholm integral equations. These computations, however, had to be performed through access to the High Performance Computing facilities at Compute Canada.

5. Numerical results

Using the numerical procedure outlined previously, we present the numerical results for the poroelastic response of a rigid circular indenter subjected to an axial loading in the form of a Heaviside step function as $P_3 H(t)$. The Degree of Settlement of the rigid circular indenter is of particular interest in engineering applications and can be defined as

$$U = \frac{\Delta (t) - \Delta (t_0)}{\Delta (t_\infty) - \Delta (t_0)} \quad (63)$$

In this expression, $\Delta (t)$ is the displacement of the rigid indenter at time $t$; $\Delta (t_0)$ and $\Delta (t_\infty)$ are, respectively, the initial and final displacements for the rigid circular indenter. The final displacement of the rigid circular indenter, $\Delta (t_\infty)$ corresponds to the solution for Boussinesq’s problem of a purely elastic halfspace (see Eqs. (i) and (ii)), while the initial value $\Delta (t_0)$ corresponds to the displacement of a rigid circular disc indenting an incompressible elastic halfspace ($\nu \rightarrow 1/2$).

The analytical solution presented in this paper is used to calibrate the accuracy of computational schemes that can be used to examine contact problems with more sophisticated geometries and boundary conditions and do not lend themselves to formulations based on analytical approaches. The computational modelling of the contact problem, in particular, can be influenced by several factors including, (i) the extent of the domain used to model the semi-infinite region, (ii) the mesh discretization adopted to accommodate sharp stress gradients, (iii) the singular stress fields that can be present at discontinuous boundary conditions applied to the displacement, stress and pore pressure fields and (iv) the ability of the computational scheme to accommodate the limit of material incompressibility as $t \rightarrow 0$. To the authors’ knowledge, there are no commercially available or public domain computational treatments that will simultaneously address the items (i) to (iv) identified previously. A complete computational treatment of the contact problem addressing all of the above items is also beyond the scope of this paper. The basic objective is to examine the extent to which a conventional finite element code that has limited capabilities can provide computational solutions to the poroelastic contact problem. By far the main limitation of computational treatments is the inability to produce accurate results for the case as $t \rightarrow 0$ when the poroelastic medium is an incompressible elastic solid. Unless the finite element approach has a mixed formulation (see e.g. Herrmann, 1965; Lee, Mardal & Winther, 2017; Phillips & Wheeler, 2009; Zienkiewicz & Shiomi, 1984) embedded in the solution scheme, the limit of incompressibility is unattainable. This limitation leads to constraints on the mesh refinement and semi-infinite domain modelling through the use of conventional approaches. The mesh refinement aspects can be addressed through suitable adaptive schemes and the modelling of semi-infinite regions requires the implementation of infinite elements (Bettess, 1977; Karapurupu, Selvadurai & Tanoeaodibjo, 1990; Selvadurai & Karapurupu, 1989; Simoni & Schrefler, 1987).
Since the computational modelling only relates to the evaluation of global results such as the load-displacement relationship, the incorporation of singular elements is not contemplated. At the outset there is an exact closed form solution for the load-displacement relation of the adhesive indentation of the rigid indenter for $t \to 0$ when $\nu \to 0.5$. From the result (ii), $(P/8Ga\Delta)_{\text{analytical}} = 1$. This result can be used to develop the mesh refinement and boundary locations of the domain to arrive at a computational solution with an acceptable accuracy.

In this paper, the computational solution to the contact problem is obtained using the finite element-based multiphysics simulation software COMSOL™. Fig. 2 shows the finite element representation of the model and the boundary conditions used to simulate the contact problem. The exterior boundaries of the domain are chosen sufficiently remote from the contact region, ensuring that the far-field boundary conditions are satisfied [i.e. the discretized domain includes the region within $r \in (0, R = 100a), z \in (0, H = 100a)$]. At the surface, the radial displacement is set to zero inside the indentation region, whereas the shear stress is considered to be zero outside this region. The mesh consists of 942,080 triangular axisymmetric elements with three degrees of freedom. A graded mesh was created in the domain, with finer meshes close to the contact region having a maximum element size of 0.01 $a$ and maximum element growth rate of 1.05. Extra mesh refinements were carried out to ensure mesh independence of the computational results and for establishing a computational model domain that can replicate domains of semi-infinite extent. The use of infinite elements in this regard is advocated but such refinements are not employed in the COMSOL™ modelling adopted for providing the comparisons. However, the COMSOL™ code has the capability to examine the elastic contact problem for an isotropic incompressible elastic material. The computational solution for a nearly incompressible elastic solid with $\nu = 0.499$ gives $[P(1 - \nu)/4Ga\Delta]_{\text{FE elastic}} = 1.01446$ and the corresponding analytical solution gives $[P(1 - \nu)/4Ga\Delta]_{\text{analytical}} = 1$. Also, as $t \to \infty$, the solution for the poroelasticity problem will correspond to the “end of consolidation” state with the skeletal value of Poisson’s ratio and shear modulus. For an elastic solid with $\nu = 0$, the exact analytical result gives $[P(1 - \nu)/4Ga\Delta]_{\text{analytical}} = 1.0986$ and the corresponding computational results give $[P(1 - \nu)/4Ga\Delta]_{\text{FE elastic}} = 1.10834$. The computational results for elastic solids indicate that the basic computational procedures provide satisfactory results for modelling the indentation. Table 1 compares the initial and final values of the consolidation settlement with the elasticity solutions. Results show that the embedded computational scheme is unable to accommodate the limit of material incompressibility at the initial stage of consolidation but it is capable of estimating the settlement over larger time scales. The limiting cases for $t \to 0$ and $t \to \infty$ of the closed form poroelasticity solution agree with the elasticity solution (Eq. (ii)) to within an accuracy of 0.00015%.

The drainage conditions on the surface of the halfspace within the contact zone and exterior to it correspond to either completely permeable (Case I), completely impervious (Case II), or partially permeable (Case III). The effect of drainage on the Degree of Settlement of a rigid circular indenter in frictionless contact with a saturated poroelastic halfspace is plotted...
against the logarithm of the non-dimensional time factor ($t^* = ct/a^2$) in Fig. 3. These results are in exact agreement with the results obtained by Yue and Selvadurai (1995a).

Fig. 4 shows the influence of drainage conditions on the Degree of Settlement of a rigid circular indenter in adhesive contact with a poroelastic halfspace. As is evident, the results associated with the partially permeable case (Case III) are bounded by the upper values associated with the entirely permeable surface (Case I), and the lower values associated with the entirely impervious surface (Case II). The Degree of Settlement for the partially permeable halfspace (Case III) approaches those of Case I at the final stages of consolidation at around $t^* > 1000$, and approaches those of Case II at the initial stages of consolidation at around $t^* \leq 0.001$. The corresponding computational results are shown in blue for all three cases. A comparison between the computational and the analytical results for two sets of Poisson’s ratios are presented in Table 2. For all three cases the error substantially decreases for $t^* \geq 1$ and becomes less than 2%.

By introducing a displacement ratio defined as $\bar{U}_2 (= \Delta(t)/\Delta(t_0))$ for both frictionless and adhesive contact, we can replot the results, as shown in Fig. 5. From these curves we conclude that the displacement ratio for frictionless contact is larger for all three drainage conditions for $t^* > 1$, and is approximately 10% larger at the final stages of consolidation.

Fig. 6 plots the displacement ratio $\bar{U}_2$ against the non-dimensional time factor $t^*$ for different values of Poisson’s ratio. The upper bound for the displacement ratio is associated with $\nu = 0$ and the lowest bound is associated with the incompressible case, which is approximately equal to 1 throughout the entire time range for partially drained frictionless and adhesive contact.

In Fig. 7 the displacement ratio $\bar{U}_2$ is plotted against the non-dimensional time factor $t^*$ for different values of Poisson’s ratio and drainage boundary conditions. The shaded regions define the extent by which results for frictionless and adhesive contacts can change.
Table 2
Comparison of the analytical and computational results for the Degree of Settlement $U$ of a rigid circular disc in adhesive contact (boundary conditions corresponding to Case-I - III (A) are stated in Section 3.1).

| $\nu$ | $t^*$ | Case-I (A) | | Case-II (A) | | Case-III (A) |
|-------|-------|------------|----------------|------------|----------------|
|       |       | $U_{\text{Analyt}}$ | $U_{\text{FE}}$ | Error% | $U_{\text{Analyt}}$ | $U_{\text{FE}}$ | Error% | $U_{\text{Analyt}}$ | $U_{\text{FE}}$ | Error% |
| 0.0   | 0.0001| 0.0293     | 0.0260         | 11.31   | 0.0044 | 0.0042         | 5.02     | 0.0070 | 0.0064 | 8.70 |
| 0.01  | 0.1824| 0.1728     | 5.25           | 0.0593 | 0.0552 | 7.00     | 0.0880 | 0.0816 | 7.19 |
| 1     | 0.6616| 0.6536     | 1.20           | 0.4734 | 0.4639 | 1.99     | 0.6126 | 0.6026 | 1.62 |
| 100   | 0.9501| 0.9441     | 0.63           | 0.9168 | 0.9093 | 0.74     | 0.9484 | 0.9419 | 0.68 |
| 10,000| 0.9956| 0.9922     | 0.41           | 0.9907 | 0.9828 | 0.35     | 0.9950 | 0.9926 | 0.37 |
| 0.25  | 0.0001| 0.0358     | 0.0325         | 9.28   | 0.0060 | 0.0058     | 3.18    | 0.0098 | 0.0089 | 9.48 |
| 0.01  | 0.2141| 0.2069     | 3.36           | 0.0707 | 0.0674 | 4.26     | 0.1058 | 0.1003 | 5.18 |
| 1     | 0.7146| 0.7086     | 1.08           | 0.5261 | 0.5160 | 1.92     | 0.6671 | 0.6596 | 1.13 |
| 100   | 0.9691| 0.9616     | 0.77           | 0.9326 | 0.9267 | 0.63     | 0.9681 | 0.9608 | 0.74 |
| 10,000| 0.9963| 0.9920     | 0.43           | 0.9923 | 0.9925 | 0.38     | 0.9959 | 0.9923 | 0.40 |

Fig. 4. Effect of drainage condition on the Degree of Settlement of a rigid circular disc in adhesive contact (A) with a saturated poroelastic halfspace.

Fig. 5. Effect of contact and drainage condition on the displacement ratio of a rigid circular disc on a saturated poroelastic halfspace (A: adhesive contact; F: frictionless contact).
Fig. 6. Effect of Poisson’s ratio on the displacement ratio of a rigid circular disc on a saturated poroelastic halfspace: Case III (frictionless (F) and adhesive (A) contact).

Fig. 7. Effect of Poisson’s ratio and drainage condition on the displacement ratio of a rigid circular disc on a saturated poroelastic halfspace (A: adhesive contact; F: frictionless contact).

Fig. 8. Effect of time on the interface pressure for a rigid circular disc on a saturated poroelastic halfspace: Case III (frictionless (F) and adhesive (A) contact).
Fig. 8 presents the non-dimensional interface pressure at different times for frictionless and adhesive partially permeable contact. This figure shows that the pore fluid pressures at the contact zone are larger for the adhesive contact compared to the frictionless contact at different times. It can be seen that there is an appreciable change in the pore fluid pressure distribution during the initial stages of consolidation, particularly in regions close to the edge of the indenter. The pore fluid pressure distribution, however, is more uniform beneath the indenter at larger values of ct/α².

6. Conclusions

The paper presents an in-depth study of the problem of the axisymmetric indentation of a poroelastic halfspace with a circular planform where consistent boundary conditions are prescribed on the contact conditions in terms of the adhesion and pore water pressure boundary conditions. Attention is focused on the estimation of the time-dependant rigid body displacement of the indenter, which can be evaluated through the numerical solution of the sets of simultaneous Fredholm integral equations of the second kind and the numerical inversion of the associated Laplace transforms. The approach adopted omits any oscillatory form of the singularities that are associated with effective stresses at the edges of the cylindrical indenter when the boundary conditions change from completely bonded to traction free and when the pore fluid pressures exhibit a similar change from Dirichlet to Neumann boundary conditions. Despite these singularities, the time-dependant force resultants can be accurately evaluated to generate estimates for the degree of consolidation settlement. The limiting cases for t → 0 and t → ∞ agree with exact closed form solutions to within an accuracy of 0.00015%. The numerical results given in this study indicate that the resulting displacement at different stages of consolidation is greater for the frictionless contact compared to the adhesive contact assumption for all three drainage conditions. Moreover, the Poisson’s ratio has proven to have a strong influence on the magnitude and rate of consolidation. In the limiting case of the incompressible poroelastic solid, the consolidation effect is absent since no excess pressure can be developed within the material and thus the medium exhibits only elastic deformations.

Declaration of Competing Interest

We have no relevant interest(s) to disclose.

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Appendix A

The kernel functions \( \tilde{\Omega}^i_{jk}(r, \rho, s); \) (\( i = 1, II, III; \) \( j, k = 1, 2, 3 \)) can be written as

\[
\tilde{\Omega}^i_{11}(r, \rho, s) = \frac{2}{\pi} \int_0^\infty \tilde{\Omega}^i_{1}(\xi, s) \cos(\xi r) \cos(\xi \rho) d\xi; \quad i = I, II
\]

\[
\tilde{\Omega}^i_{12}(r, \rho, s) = \frac{1}{\pi} \int_0^\infty \left[ \tilde{\Omega}^i_{2}(\xi, s) + 1 \right] \sin(\xi r) \sin(\xi \rho) d\xi; \quad i = I, II
\]

\[
\tilde{\Omega}^i_{21}(r, \rho, s) = \frac{1}{\pi} \int_0^\infty \left[ \tilde{\Omega}^i_{2}(\xi, s) + 1 \right] \sin(\xi r) \cos(\xi \rho) d\xi; \quad i = I, II
\]

\[
\tilde{\Omega}^i_{22}(r, \rho, s) = \frac{2}{\pi} \int_0^\infty \tilde{\Omega}^i_{2}(\xi, s) \sin(\xi r) \sin(\xi \rho) d\xi; \quad i = I, II
\]

\[
\tilde{\Omega}^i_{11}(r, \rho, s) = \frac{2}{\pi} \int_0^\infty \tilde{\Omega}^i_{1}(\xi, s) \cos(\xi r) \cos(\xi \rho) d\xi;
\]

\[
\tilde{\Omega}^i_{12}(r, \rho, s) = \frac{1}{\pi} \int_0^\infty \left[ \tilde{\Omega}^i_{2}(\xi, s) + 1 \right] \cos(\xi r) \sin(\xi \rho) d\xi;
\]

\[
\tilde{\Omega}^i_{13}(r, \rho, s) = \frac{1}{\pi} \int_0^\infty \left[ \tilde{\Omega}^i_{2}(\xi, s) + \xi^{-1} \right] \sin(\xi r) \cos(\xi \rho) d\xi;
\]

\[
\tilde{\Omega}^i_{21}(r, \rho, s) = \frac{1}{\pi} \int_0^\infty \left[ \tilde{\Omega}^i_{2}(\xi, s) + 1 \right] \sin(\xi r) \cos(\xi \rho) d\xi;
\]

\[
\tilde{\Omega}^i_{22}(r, \rho, s) = \frac{2}{\pi} \int_0^\infty \tilde{\Omega}^i_{2}(\xi, s) \sin(\xi r) \sin(\xi \rho) d\xi;
\]

\[
\tilde{\Omega}^i_{23}(r, \rho, s) = \frac{1}{\pi} \int_0^\infty \xi^{-1} \left[ \tilde{\Omega}^i_{2}(\xi, s) + 1 \right] \sin(\xi r) \sin(\xi \rho) d\xi;
\]

\[
\tilde{\Omega}^i_{23}(r, \rho, s) = \frac{1}{\pi} \int_0^\infty \xi^{-1} \left[ \tilde{\Omega}^i_{2}(\xi, s) + 1 \right] \sin(\xi r) \sin(\xi \rho) d\xi;
\]

\[
\tilde{\Omega}^i_{23}(r, \rho, s) = \frac{1}{\pi} \int_0^\infty \xi^{-1} \left[ \tilde{\Omega}^i_{2}(\xi, s) + 1 \right] \sin(\xi r) \sin(\xi \rho) d\xi;
\]
\[
\Omega^{(1)}_{12}(r, \rho, s) = -\frac{2}{\pi} \int_0^\infty \xi \tilde{\hat{\Phi}}_3^{(1)}(\xi, s) \sin(\xi \rho) \cos(\xi \rho) d\xi;
\]

(A.11)

\[
\Omega^{(2)}_{32}(r, \rho, s) = \frac{1}{\pi \eta} \int_0^\infty \xi \tilde{\hat{\Phi}}_3^{(2)}(\xi, s) \sin(\xi \rho) \eta \sin(\xi \rho) d\xi;
\]

(A.12)

\[
\Omega^{(3)}_{23}(r, \rho, s) = \frac{2}{\pi} \int_0^\infty \xi \tilde{\hat{\Phi}}_3^{(2)}(\xi, s) \sin(\xi \rho) \sin(\xi \rho) d\xi;
\]

(A.13)

where

\[
\tilde{\hat{\Phi}}_1 = \frac{\xi - \varphi}{2\eta} \quad \text{and} \quad \tilde{\hat{\Phi}}_2 = \frac{2\xi^2 - \xi \varphi - \varphi^2}{2(\xi^2 - \eta \xi \varphi - \eta \varphi^2)}; \quad i = I, III
\]

(A.14)

\[
\tilde{\hat{\Phi}}_3 = -\frac{(\xi + \varphi)}{\xi} \tilde{\hat{\Phi}}_2
\]

(A.15)

References


