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Research Paper

Irreversibility of soil skeletal deformations: The Pedagogical Limitations of Terzaghi's celebrated model for soil consolidation

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ABSTRACT

The paper examines the role of irreversibility of skeletal deformations of a consolidating soil and its role in the pedagogy for explaining the development of pore fluid pressures in saturated soils subjected to loading and unloading. The role of irreversibility is demonstrated through the consideration of a one-dimensional consolidation model with an elasto-plastic skeletal response.

1. Introduction

The treatise by Terzaghi (1923) is widely recognized as the birth of the discipline of soil mechanics (see also Terzaghi and Fröhlich, 1936) and it is recognized as a commanding publication that brought to attention the need to elevate the topic of the mechanics of fluid-saturated soils from empiricism to a discipline firmly based on scientific principles. The controversies and the tragic consequences associated with Fillunger (1936) following Terzaghi's publication are presented in a comprehensive historical study of developments in porous media written by de Boer (2000). As remarked by Taylor and Merchant (1940) in their study of the process of secondary consolidation, "*The theory of consolidation developed by Dr. Karl von Terzaghi is one of the most important theoretical contributions, which have been made in the new branch of Foundation Engineering known as Soil Mechanics*". A comprehensive documentary of the history and development of the theory of consolidation is given in a number of publications including the volume by de Boer (2000) and the extensive bibliographies are provided by Florin (1948), Paria (1963), Zaretskii (1972), Schiffman (1984), Selvadurai (1979, 2007), Lewis and Schrefler (1998), Coussy (1995), Aleynikov (2011), Schrefler (2002), Cheng (2015), Selvadurai and Suvorov (2016a) and Selvadurai and Samea (2021). Terzaghi's contribution to the development of the one-dimensional theory of consolidation introduces several assumptions including: the complete saturation of the pore space, incompressibility of the pore fluid and the material composing the porous skeleton, the partitioning of the external stresses to skeletal stresses and pore fluid pressures, the validity of the linear form of Darcy's law for the entire range of the consolidation process, the linear relationship between the stresses in the porous skeleton and the corresponding skeletal deformations (no explicit mention of *reversibility* of skeletal deformations) and the added observation that the time lag in the consolidation is entirely due to the low permeability of the soil. A mechanical model for replicating the consolidating soil was first presented by Terzaghi (1927). It consisted of a fluid filled container with spring elements and narrow apertures in the movable platen representing the pores of the soil as shown in the idealization in Fig. 1. The requirements for low permeability to ensure time dependency of the consolidation process is also mentioned on page 266 of Terzaghi (1943). The mechanical model of a consolidating soil was also presented by Terzaghi and Fröhlich (1936), where the single mechanical model was replaced by the "Mehrkolbenmodell" or the "Many Piston model" (Fig. 2).

Terzaghi and Peck (1948) also presented the "Mehrkolbenmodell" (Fig. 3) with schematic expositions of the pore pressure distributions over the thickness of the consolidating element. Surprisingly, there is no record of the mechanical analogue in the volumes by Terzaghi (1923, 1943) although the analogue is ubiquitous in nearly every undergraduate textbook devoted to soil mechanics. Examples of these can be found in the volumes by Taylor (1948), Krynine (1947) and Tschebotarioff (1951) and these are reproduced in Figs. 4, 5 and 6, respectively. The volumes attribute these mechanical analogues to works of Terzaghi (1923) and Terzaghi (1927), respectively. References to Terzaghi's developments are also cited by Scott (1963) and Harr (1966) and the latter refers to Terzaghi (1924) as a source for the introduction of the theory of

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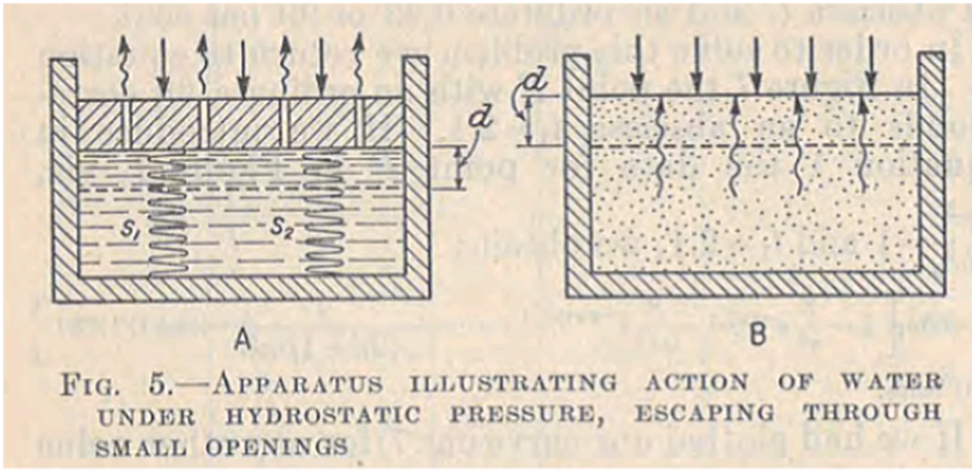

Fig. 1. The mechanical model of a consolidating soil first presented by Terzaghi (1927).

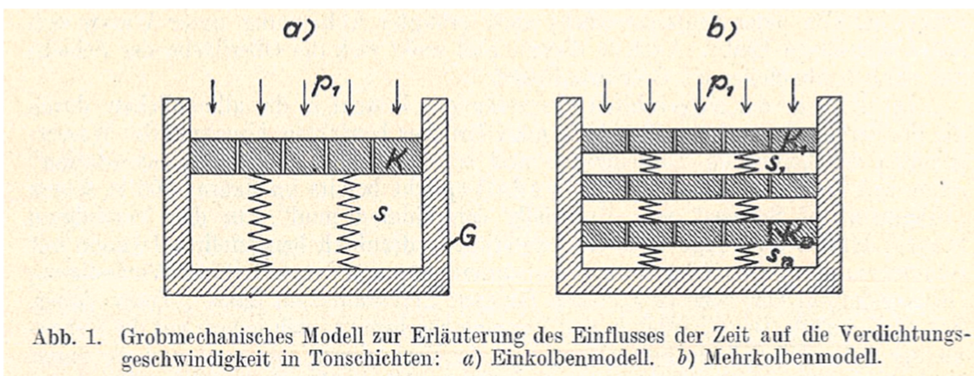

Fig. 2. Mechanical models of a consolidating soil presented by Terzaghi and Fröhlich (1936).

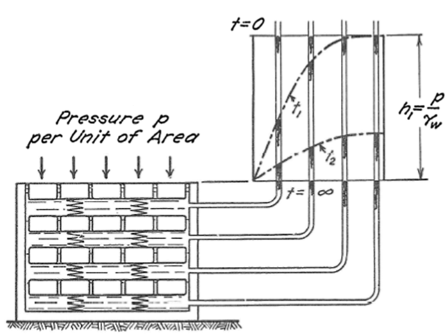

FIG. 27. Device for demonstrating mechanics of process of consolidation.

Fig. 3. The mechanical model used to demonstrate soil consolidation (After Terzaghi and Peck, 1948).

soil consolidation.

At this stage it is useful to highlight important attributes related to Terzaghi's one-dimensional theory. The first is the theory of effective stress, which states that the total stress (σ_{ij}) is partitioned between the stress carried by the porous skeleton (σ'_{ij}) and the isotropic pressure (u) carried by the pore fluid, i.e.

$$\sigma_{ij} = \sigma'_{ij} + u\delta_{ij} \tag{1}$$

where δ_{ij} is Kronecker's delta function. The historical aspects of the theory of effective stresses in soils have differing interpretations. Sir Alec Skempton (1960) attributes the first reflections of the concept to Sir Charles Lyell in 1871, then by J. Boussinesq in 1876 and by Sir Osborne Reynolds in 1886. According to Skempton (1960), the idea of effective stress was formulated in a general form by Terzaghi (1936) and the concept introduced by Terzaghi (1923) is the one-dimensional version $\sigma' = \sigma - u$, which formed the basis for the development of the one-dimensional theory of consolidation. The work of Biot (1935), however, preceded the extension to three-dimensions proposed by Terzaghi in 1936. Terzaghi (1936) postulated that "*The stresses in any point of a section through a mass of soil can be computed from the total principal stresses σ_1, σ_2, σ_3 which act in this point. If the voids of the soil are filled with water under a stress u, the total principal stresses consist of two parts. One part,u, acts in the water and in the solid in every direction with equal intensity. It is called the neutral stress* (*or the porewater pressure*). *The balance $\sigma'_1 = \sigma_1 - u$, $\sigma'_2 = \sigma_2 - u$, and $\sigma'_3 = \sigma_3 - u$ represents an excess over the neutral stress u and it has its seat exclusively in the solid phase of the soil. This fraction of the total principal stresses will be called the effective principal stresses…*" An informative record of the stages in the development of the theory of effective stresses is also given by de Boer and Ehlers (1990).

The tensorial representation (1) is a direct extension of the one-dimensional scalar partitioning presented in Terzaghi's 1923 studies. The main observation is that in the partitioning process, neither the constitutive relationships of the porous skeleton nor the constitutive relationships of the pore fluid are present. This is, of course, at variance with other stress partitioning processes in multiphasic materials where the constitutive responses are expected to influence the mechanics of stress partitioning. A simple example of this is the two-bar elastic composite element extensively employed in undergraduate Solid Mechanics courses (Timoshenko, 1930). Also, the partitioning process has

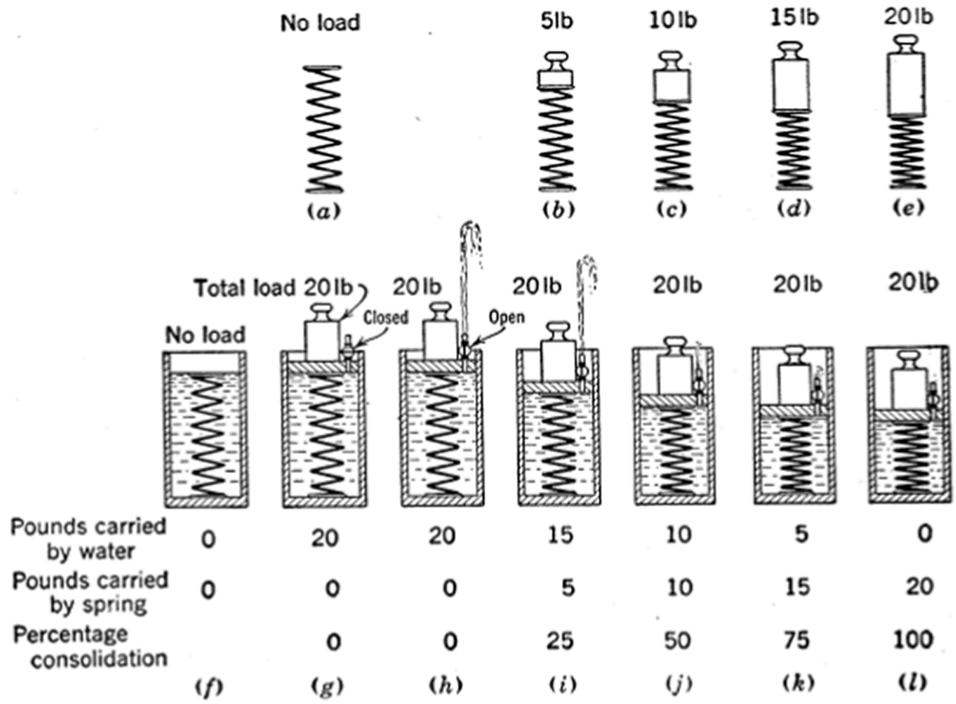

FIG. 10·8 The piston and spring analogy.

Fig. 4. The mechanical model used to demonstrate soil consolidation (After Taylor, 1948).

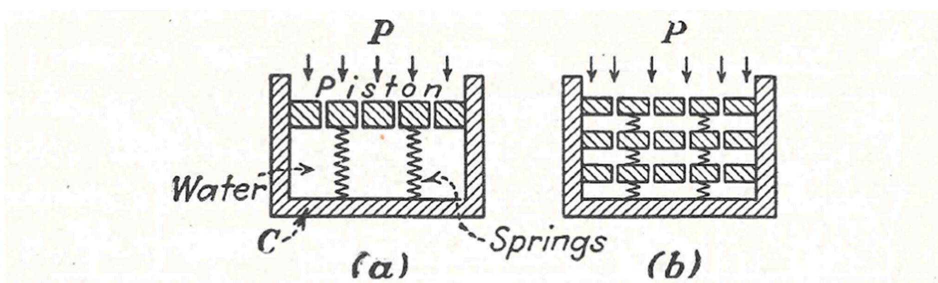

FIG. 6:3.—Mechanical model of consolidation. (*After Terzaghi.*)

Fig. 5. The mechanical model used to demonstrate soil consolidation (After Krynine, 1947).

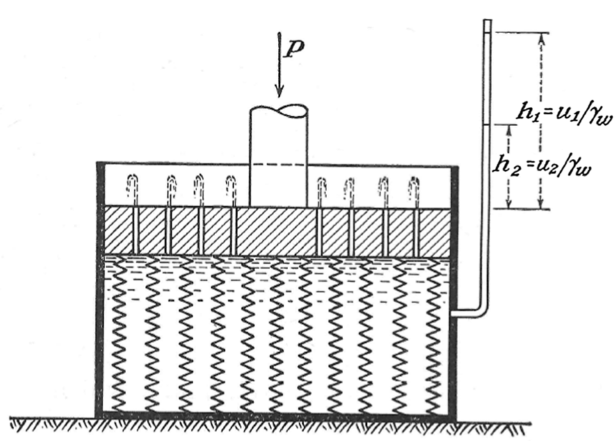

FIG. 6-1. Model of the type used by Terzaghi, Ref. 350, 1927, to explain by analogy the process of consolidation of a saturated clay.

Fig. 6. The mechanical model used to demonstrate soil consolidation (After Tschebotarioff, 1951).

its antecedents in the works of Kelvin (1878), Voigt (1910) and others. The concept nonetheless was readily adopted as an important precept in the development of soil mechanics, particularly in relation to the study of the role of pore fluid pressures in geotechnical problems. The second important concept related to the development of Terzaghi's theory of consolidation is the assumption of the linear behaviour of the stress-strain response of the soil skeleton. Terzaghi and co-workers (Terzaghi and Peck, 1948; Art 13) were well aware of the limitations of linear elasticity and reversibility in describing the deformability characteristics of the soil skeleton but chose to adopt a linear relationship (i.e. a spring element) between the changes to the void ratio with effective stress. All approaches dealing with the estimation of consolidation settlements were deeply rooted in semi-logarithmic relationships between the void ratio and the effective stress and irreversible processes were implicit in the approaches that introduced concepts such as normally-consolidated and over-consolidated clays that adopted ingenious approaches for determining the pre-consolidation stress. If the semi-logarithmic relationship was used to describe the skeletal deformability, the resulting governing equation for the one-dimensional consolidation equation would have been an unwieldy *non-linear partial differential equation* and the effect of irreversible deformations would have resulted only in numerical approaches to the solution of the resulting consolidation equation. Such procedures were not in vogue at the time of development of the consolidation theory. The prudent linear approach advocated by Terzaghi (1923, 1943) resulted in the classical consolidation equation, which is a *second-order linear partial differential equation* of the parabolic-type, which has been extensively investigated in connection with Fourier heat conduction and other diffusive phenomena (Carslaw and Jaeger, 1959; Crank, 1970; Hill and Dewynne, 1987; Lamb, 1995; Selvadurai, 2000). The assumptions of linearity in the void ratio vs. the effective stress relationship and the absence of irreversibility in a *loading-unloading* cycle place constraint in examining the development of pore fluid pressures during a loading excursion that involves both loading and unloading.

Terzaghi's theory, which describes the one-dimensional theory of the consolidation for a fluid saturated soil with a purely elastic porous skeleton, was extended to three-dimensions by Rendulic (1936) but the developments are considered to be incomplete. The complete development of the three-dimensional theory of poroelasticity, which deals with the mechanics of a fluid-saturated porous elastic skeleton with an incompressible pore fluid was presented by Biot (1935, 1941). The theory represents a canonical development of the three-dimensional theory of consolidation, which has also been presented in subsequent independent studies by Mandel (1950, 1953) and others. In modern expositions of Biot's theory, appeal is made to the theory of mixtures extensively developed in continuum theories applied to multiphasic fluid-saturated media (see e.g. Mills, 1966; Green and Naghdi, 1965, 1967; Bowen and Wiese, 1969; Atkin and Craine, 1976; Bowen, 1976; Ehlers, 1991; de Boer 2000). It is also important to note that the application of mixture theories is permissible if the porous region is fully saturated with a single fluid. In the case of unsaturated media, the presence of multi-phasic fluids (liquids and gases) together with interphasic stress phenomena, such as surface tension forces, requires the introduction of additional measures to correctly describe the concept of effective stress.

Terzaghi (1943; Art. 106)) presents a commentary on Biot's contribution as follows: "*All these investigations were based on the assumption that the coefficient of consolidation c_v contained in equation (1)* [Which is the direct extension of the one-dimensional equation to its three-dimensional Laplacian operator based equation] *is a constant. For processes of consolidation involving linear flow this assumption is known to be reasonably accurate. However, in connection with two- and three-dimensional processes of consolidation, the same assumption should be regarded as a potential source of errors whose importance is not yet known. Biot also assumed that c_v has the same value for both compression and swelling. This assumption is never justified. A better approximation could be obtained by assuming that c_v for swelling is equal to* ***infinity***". Terzaghi provides a critique of Biot's developments but omits reference to his own one-dimensional theory where the same criticism can be applied if a loaded consolidating region is unloaded. His appreciation of *irreversibility of deformations in the porous skeleton* is, however, very clear. The method for incorporating the differences in stiffness during loading and unloading is not that straightforward and the author would suspect that a purely analytical result for generalized three-dimensional stress states is not feasible. If the stress state involves only the principal stresses and if monotonic loadings ensure that the principal stresses follow either a compressive or tensile loading path, then the concept of bi-modularity can be invoked to develop an analytical solution. The concept of a bi-modular elastic material has been introduced in connection with the study of elastic solids (Timoshenko, 1930) and in the study of the mechanics of elastic materials reinforced with fibres (Schwartz and Schwartz, 1967; Ambartsumian, 1966; Mkrtichian, 1970; Jones, 1977; Green and Mkrtichian, 1977; Spence and Mkrtchian, 1977; Bert 1977). In the context of geologic media containing randomly distributed cracks, the elastic moduli in tension will be different from the elastic modulus in compression if the stress states cause crack closure. The radial expansion of a spherical cavity in a poroelastic bi-modulus material was examined by Selvadurai and Mahyari (1998). In this case, the stress state is spherically symmetric, which allows for the formulation via a bi-modulus approach.

Although Biot's theory of poroelasticity is widely applied to examine the consolidation of saturated clays (McNamee and Gibson, 1960a,b; Gibson and McNamee, 1963; Agbezuge and Deresiewicz, 1974, 1975; Chiarella and Booker, 1975; Rajapakse and Senjuntichai, 1993; Detournay and Cheng, 1993; Yue and Selvadurai, 1995; Lan and Selvadurai, 1996; Selvadurai, 1996; 2007; Verruijt, 2015; Cheng, 2015; Selvadurai and Kim, 2016; Selvadurai and Samea, 2021, and others) the theory finds a firmer basis when applied to fluid-saturated rocks (Rice and Cleary, 1976; Rice, 1992; Souley et al., 2015; Cheng, 2015; Selvadurai and Suvorov, 2016a; Selvadurai and Najari, 2017; Chen et al., 2018; Zhang et al., 2019 and others) that can maintain their elastic behaviour over a significant stress range, although the theory has also found applications even in the modelling of Callovo-Oxfordian claystone (Braun et al., 2019). An important development in Biot's theory is the modification of the effective stress equation to the form

$$\sigma_{ij} = \sigma'_{ij} + \alpha u \delta_{ij} \tag{2}$$

where α is the Biot coefficient represented by

$$\alpha = 1 - \frac{K_D}{K_S} \tag{3}$$

and K_D and K_S are, respectively, the bulk modulus for the porous skeleton void of any pore fluid and the bulk modulus of the solid material composing the porous skeleton. If the bulk modulus of the solid material composing the porous skeleton is much greater than the bulk modulus of the porous skeleton, then $(K_D/K_S) \to 0$ and (2) reduces to Terzaghi's result (1). The measurement of the Biot coefficient for rocks in particular is straightforward if the geomaterial has a high permeability, which facilitates saturation of the pore space (see e.g. Rice and Cleary, 1976; Berryman, 1992; Wang, 2000; Mavko et al., 2009). With very low permeability rocks, the saturation procedure required for estimating K_S is unreliable and the presence of trapped air can give rise to erroneous estimates of K_S and attention is usually focused on the application of multiphasic approaches for its estimation. Examples of these are given by Selvadurai (2019) and Selvadurai et al. (2019). It can also be shown that other factors, such as pore shape, can influence the Biot coefficient (Selvadurai and Suvorov, 2020).

Of related interest are the studies by Skempton (1954) and Bishop (1973) that implicitly use elasticity concepts when developing the pore-pressure parameters, in the sense that irreversibility and loading path dependency are not addressed in these studies. The work of Pande and

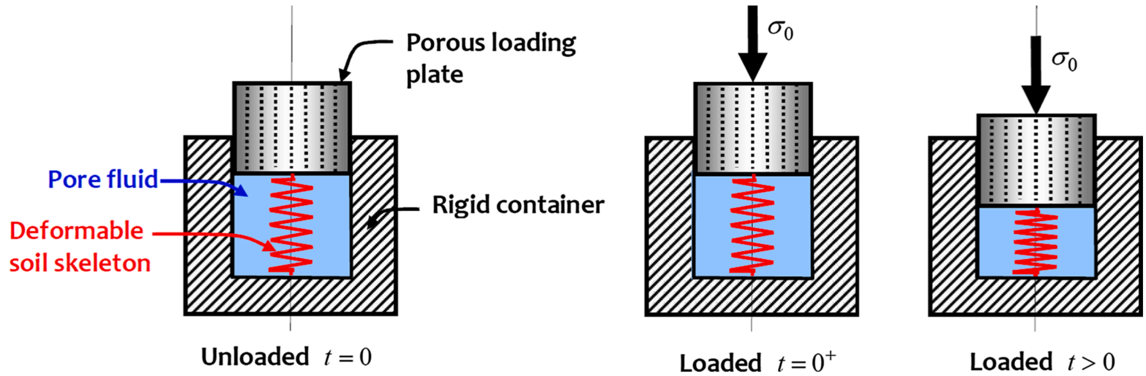

Fig. 7. Terzaghi analogue for the consolidation process.

Pietruszczak (1990) is one of the early studies where the issue of failure of the soil skeleton is discussed in relation to the estimation of the pore pressure parameters. Recently, Suvorov and Selvadurai (2019) have examined the influence of elasto-plasticity of the porous skeleton on the Biot coefficient.

2. Irreversibility in the Skeletal Deformability

The issue of irreversible deformations of geomaterials such as soils, regardless of the level of saturation, is universally accepted. Idealized skeletal responses based on elastic reversibility are rare and can be both used and abused. The a priori assignment of stress levels that can maintain certain soils in a condition that minimizes irreversible phenomena is advocated (Terzaghi, 1943; Taylor, 1948; Scott, 1963; Harr, 1966; Suklje, 1969; Gibson, 1974; Burland, 1989; Davis and Selvadurai, 1996; Selvadurai, 2007) and an incorrect setting can lead to erroneous interpretations. Terzaghi was well aware of the need to incorporate effects of irreversibility into his fundamental developments in the theory of consolidation, but addressing this issue was far from straightforward. Developments in the computational aspects of soil mechanics were at their infancy and the studies by Taylor and Merchant (1940), which incorporated viscoelastic elements to represent the constitutive behaviour of the porous skeleton to account for secondary consolidation, is probably an early development of addressing irreversible phenomena albeit in a time-dependent setting rather than in a time-independent context (see also, Geuze and Tan, 1950; Biot, 1956; Gibson and Lo, 1961; Kravtchenko and Sirieys, 1966; Schiffman et al., 1966; Ichikawa and Selvadurai, 2012, to name a few). Terzaghi's theory of one-dimensional consolidation gives rise to the second-order partial differential equation

$$c_v \frac{\partial^2 u}{\partial z^2} = \frac{\partial u}{\partial t} \tag{4}$$

where

$$c_v = \frac{k(1+e)}{\gamma_w a_v} \tag{5}$$

where k is the hydraulic conductivity, e is the void ratio, $a_v (= -\partial e/\partial \sigma')$ is the coefficient of compressibility and γ_w is the unit weight of water. The extension of (4) to three-dimensions, proposed by Rendulic (1936) simply replaces the spatial derivative in (4) by the dyadic Laplace's operator:

$$c_v \nabla^2 u = \frac{\partial u}{\partial t} \tag{6}$$

Numerous other extensions have been proposed to modify the classical theory of consolidation. For example, Zaslavsky (1964) correctly formulates Darcy's law in terms of the relative velocity between the pore fluid and the soil skeleton, which gives rise to the following equation for the equation of consolidation

$$\nabla k.\nabla\left(\frac{u}{\gamma_w}\right) = \frac{\partial}{\partial t}(1 + q_s.\nabla)[ln(1+e)] \tag{7}$$

where q_s is the average velocity of soil particles relative to a stationary frame of reference. Similarly, Mikasa (1965) formulated the consolidation equation in terms of a logarithmic function of the void ratio instead of the pore fluid pressure: i.e.

$$c_v \nabla^2 \varepsilon = \frac{\partial \varepsilon}{\partial t} \tag{8}$$

where

$$\varepsilon = ln\left(\frac{1+e_0}{1+e}\right) \tag{9}$$

Despite the noteworthy observation by Terzaghi, the irreversible aspects of the skeletal response in the context of the theory of consolidation, at least to the author's knowledge, has received scant attention. Most of the expositions of the theory of consolidation largely focused on

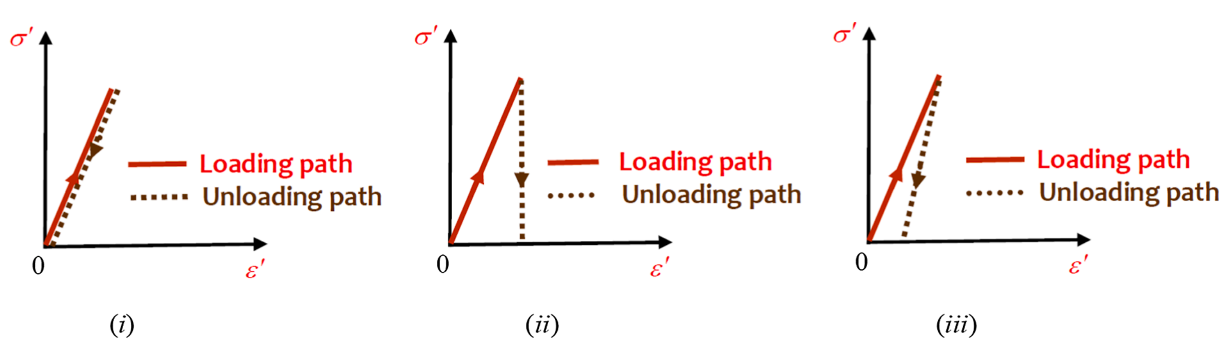

Fig. 8. One-dimensional constitutive relationship for the soil skeleton.

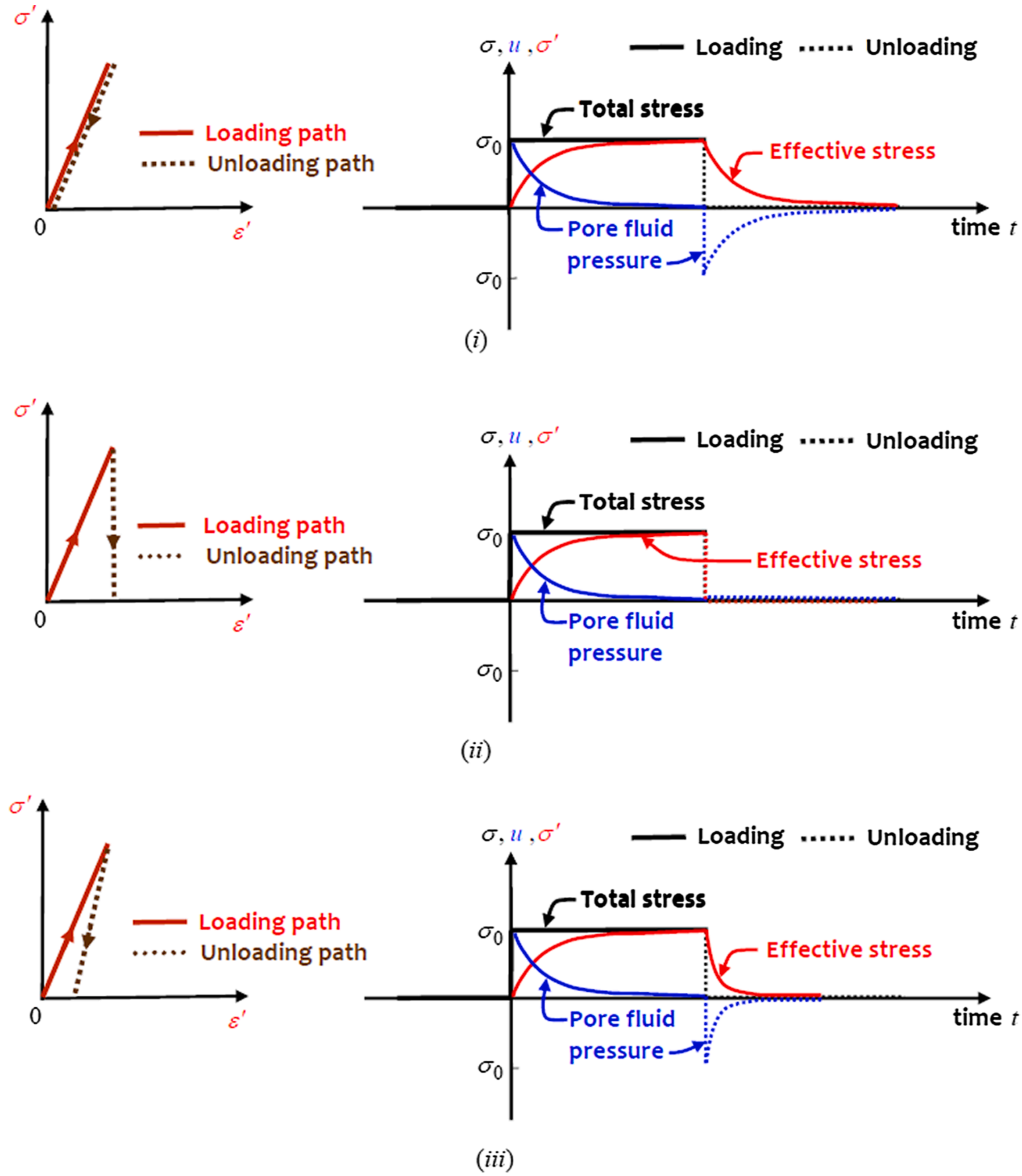

Fig. 9. Loading-unloading skeletal stress-strain relationship and the development of effective stresses and excess pore fluid pressures.

the notion advocated by Terzaghi in relation to the linearity in the relationship between effective stress and void ratio, without any mention of unloading. Any researcher who is involved in constitutive model development can appreciate the fact that the validity of a constitutive model is not proven until the response of the model taken through a cycle involving loading, unloading, stress reversal, re-loading, etc., can be correctly captured. This aspect is perhaps overlooked in many expositions adopted to model the skeletal behaviour during consolidation given in almost every elementary textbook in soil mechanics. In fact, to the author's knowledge, none of the 200-odd undergraduate texts on Soil Mechanics ever discuss the process of unloading during soil consolidation with a clear reference to the mechanism of pore pressure generation during the unloading process. This creates a pedagogical limitation when it comes to the discussion of a saturated soil such as clay during *loading* and *unloading*, and, in particular, the rationale for the pre-loading of clayey soils to minimize settlements. The situation is not drastic until the question of the unloading of a consolidating soil has to be illustrated in terms of the development of pore fluid pressures and effective stress. The one-dimensional Terzaghi model of a saturated soil is shown in Fig. 7 and attention is drawn to the *total stress* σ, the resulting skeletal or *effective stress* σ' and the *excess pore pressure* is denoted by u (the term excess pore fluid pressure is intended to identify the pore fluid pressures that are attributed to the changes in total stress). In addition, let us consider an elementary view of the one-dimensional constitutive relationship for the soil skeleton shown in Fig. 8.

Fig. 8(*i*) illustrates a skeletal deformability response that is linearly elastic, and the skeletal strain energy accumulated during a loading cycle is perfectly recoverable. Fig. 8(*ii*) illustrates a skeletal deformability response that is linear in the loading path, but the unloading path results in *zero skeletal strain recovery*. This type of skeletal behaviour was alluded to by Terzaghi in his critique of Biot's theory. Fig. 8(*iii*) illustrates a skeletal deformability response that is linear in the loading path, but the unloading takes place in a linear fashion with evidence of partial elastic recovery. These models are idealized responses intended to demonstrate the influence of the skeletal deformability response on the excess pore pressure, when the Terzaghi analogue is subjected to a loading-unloading response.

Fig. 9 (*i*) illustrates the loading-unloading response for the total stress $\sigma(t)$, the effective stress $\sigma^{'}(t)$ and the excess pore fluid pressure $u(t)$. The element is loaded by a total stress σ_0 applied as a Heaviside step function of time and the element is allowed to consolidate. As Terzaghi postulates, the pore fluid pressure will instantaneously reach the value of σ_0 and will reduce to zero as the stress is transferred to the skeleton. In this case, strain energy is stored in the consolidated soil skeleton and when the external stress is reduced to zero, to preserve overall equilibrium in the system and to satisfy the effective stress concept (1), the excess pore fluid pressure has to be negative or in a state of tension. At the moment of instantaneous unloading, the excess pore water pressure must satisfy the constraint, $u = -\sigma_0$. As time progresses, the soil will "*unconsolidate*" and the effective stresses and pore fluid pressures will reduce to zero. The concept is elementary and a simple superposition of solutions of the one-dimensional equation of consolidation for a finite or semi-infinite region with a time shift to identify the point of unloading can be used to confirm the observation. For example, for a saturated semi-infinite domain $z \in (0, \infty)$ subjected to a total stress σ_0 over the time interval $0 < t \leqslant t^*$, the time dependent distribution of pore fluid pressure in the domain is given by

$$\frac{u(z,t)}{\sigma_0} = erf\left(\frac{z}{\sqrt{4c_v t}}\right) - erf\left(\frac{z}{\sqrt{4c_v(t-t^*)}}\right) \tag{10}$$

where $erf(\Delta)$ is the error function with the series approximation

$$erf(\Delta) = \frac{2\Delta}{\sqrt{\pi}}\left(1 - \frac{\Delta^2}{1!3} + \frac{\Delta^4}{2!5} - \frac{\Delta^6}{3!7} + \ldots\right) \quad ; \Delta^2 < \infty \tag{11}$$

Fig. 9 (*ii*) illustrates the loading-unloading response for the case where the skeletal strain is completely irrecoverable. In this case, if the consolidation process is complete during the loading phase, the unloading of the externally applied stress does not result in any recovery of the skeletal strains and the excess pore fluid pressures do not exhibit any change. The consolidation deformations remain *irrecoverable*. The geotechnical engineering rationale for the pre-loading of clay layers to cause settlement is justified in these circumstances. To the author's knowledge, this elementary concept is identified in the calculation of settlements but less well explained in relation to the development of excess pore fluid pressures and the recovery of consolidation settlements. Fig. 9 (*iii*) illustrates the loading-unloading response for the case where the skeletal strains are only partly recoverable. The presence of elastic unloading will ensure that the excess pore fluid pressure generated in a fully consolidated element will become negative. The rate of decay of the effective stresses and the excess pore fluid pressure will be bounded by the responses associated with the cases illustrated in Fig. 9 (*i*) and 9 (*ii*). Other scenarios similar to those presented in these three examples can be considered and such investigations are best handled by a computational approach, where the skeletal response is modelled by appeal to a theory of plasticity capable of accommodating both reversible and irreversible skeletal strains.

3. Computational Modelling of Irreversibility in the Skeletal Deformability

The modelling of the constitutive behavior of soils, both saturated and unsaturated, that can accommodate elastic, hyperelastic, hypoelastic, elasto-plastic (with and without hardening) effects, visco-plasticity, creep, damage, etc., has been the subject of extensive research over the past century. This article is perhaps not the appropriate venue for presenting a comprehensive overview of the important developments related to the constitutive modelling of soils and geomaterial interfaces. The interested reader is referred to the seminal articles and texts by Drucker and Prager (1952), Drucker et al. (1957), Schofield and Wroth (1968), Zaretskii (1972), Desai and Siriwardane (1984), Chen and Baladi (1985), Pande et al. (1990), Darve (1990), Selvadurai and Boulon (1995), Desai (2001), Ehlers and Bluhm (2002), Willner (2003), Davis and Selvadurai (2005), Pietruszczak (2010), Selvadurai and Atluri (2010), Gens (2010), Aleynikov (2011), Puzrin (2012), Chau (2013), Wan et al. (2016), Selvadurai (2020) and others.

3.1. Poroelasto-plastic and non-linear behaviour of the soil skeleton

In extending the studies to include poroelasto-plasticity effects, we need to select an appropriate constitutive response for saturated clay-type materials. There are a variety of constitutive relations that have been proposed in the literature and for the purposes of illustration, we select an elasto-plastic skeletal response of the Modified Cam Clay type (see. e.g. Desai and Siriwardane, 1984; Davis and Selvadurai, 2005; Pietruszczak, 2010).

The study of the consolidation is best illustrated by considering the one-dimensional behaviour of a soil column of finite length (Terzaghi, 1943; Biot, 1941; Mandel, 1950, 1953, 1957; Verruijt, 2015; Cheng, 2015; Selvadurai and Suvorov, 2016b; Stickle and Pastor, 2018). The analysis can also be investigated through consideration of spherically symmetric or spherically asymmetric (de Josselin de Jong, 1953, 1957; Gibson et al., 1963; Verruijt, 2015) or other radially symmetric situations (Cryer, 1963). Similar treatments of spherical cavities and inclusions in poroelastic and damage-susceptible poroelastic media are also given by Rice et al. (1978) and Selvadurai and Shirazi (2004). The one-dimensional consolidation problem for an elastic skeletal response has been exhaustively studied in the literature and the references to these articles can be gleaned from the articles cited previously. The consideration of non-linear behaviour attributed to large strain elastic skeletal behaviour was examined in the studies by Gibson et al. (1967, 1981, 1989), Schiffman (1980) and Schiffman and Cargill (1981). The constitutive relationship adopted for the skeletal behaviour was, however, restricted to a linear model. The one-dimensional consolidation behaviour of a fluid-saturated hyper-elastic material was first examined by Selvadurai and Suvorov (2016b) where a consistent hyperelasticity model (see e.g. Rivlin, 1953, 1961; Green and Adkins, 1970; Spencer, 1970; Selvadurai and Spencer, 1972; Treloar, 1976) was used to represent the constitutive behaviour of the porous skeleton. In the studies by Selvadurai and Suvorov (2016b, 2017, 2018) and Suvorov and Selvadurai (2016), the effective stress relationship is represented by the Terzaghi (1923) result (1) rather than the Biot (1941) result since for a highly deformable hyperelastic material, K_S is assumed to be much greater than K_D. Also, the influence of hyperelastic deformations on the alteration of the fluid transport properties of the porous medium is neglected. These advances, which resulted in benchmark analytical studies were, however, restricted to the purely elastic behaviour of the porous skeleton in the absence of irreversible deformations during a cycle of loading and unloading. Along the theme of large strain behavior, computational approaches for the study of large elasto-plastic strains and strain rate sensitivity in materials including saturated soils were examined by a number of authors including Lee (1969), Small et al. (1976), Carter et al (1979), Borja and Tamagnini (1998), Borja et al. (1998), Ehlers and Bluhm (2002), Lubarda (2004) and Selvadurai and Yu (2006).

An important contribution to the analytical modelling of the elasto-plastic skeletal behaviour of the porous skeleton was made by Pariseau (1999), who examined the one-dimensional poroelasto-plastic consolidation of a saturated medium where the soil skeleton exhibits failure according to either a Mohr-Coulomb or Drucker-Prager relationship. Due to the one-dimensionality of the problem there is uncoupling of the solid deformation from the fluid flow. The fluid pressure satisfies the same form of a diffusion equation in both the elastic and elastic-plastic domains, but the diffusion coefficient is different. The solution given by Pariseau (1999) has a benchmark quality that is intended to serve as a result for validating computational approaches. The studies by Selvadurai and Suvorov (2012, 2014) focus on similar studies applied to thermo-poro-elasto-plasticity problems involving (i) boundary heating of spheres and cylinders and (ii) the separate pressurization and heating of a fluid-filled cavity.

3.2. Poroelasto-plastic behaviour of the soil skeleton under loading and unloading

In this study we consider the poroelasto-plastic behaviour of the soil skeleton, under conditions of quasi-static loading-unloading. The constitutive responses that can be adopted for the study can be many and varied but for the purposes of demonstration, we adopt a Cam-Clay model (Schofield and Wroth, 1968; Desai and Siriwardane, 1984; Davis and Selvadurai, 2005; Pietruszczak, 2010) defined by the yield function

$$(\widetilde{\sigma}-a)^2+(q/M)^2-a^2=0 \tag{12}$$

where q is the von Mises stress, a is the radius of the yield surface, $\widetilde{\sigma}$is the mean effective stress, M is the slope of the critical state line and these are defined by

$$q=\sqrt{3\widetilde{s}_{ij}\widetilde{s}_{ij}/2} \quad ; \quad \widetilde{\sigma}=-(\widetilde{\sigma}_{kk}/3); \quad \sigma_{ij}=\widetilde{\sigma}_{ij}-\alpha u\delta_{ij} \quad ; \quad \widetilde{s}_{ij}=\widetilde{\sigma}_{ij}+\widetilde{\sigma}\,\delta_{ij} \tag{13}$$

The center of the yield surface $(a,0)$ in the $(\widetilde{\sigma},q)$ plane can be expressed as $2a=\widetilde{\sigma}_c^0+\widetilde{\sigma}_c\left(\varepsilon_{kk}^{pl}\right)$, where $\widetilde{\sigma}_c^0$ is the initial yield stress for the isotropic compression stress state and

$$\widetilde{\sigma}=\widetilde{\sigma}(\varepsilon_{kk}^{pl})=\widetilde{\sigma}_c^0+\widetilde{\sigma}_c(\varepsilon_{kk}^{pl}) \tag{14}$$

is the hardening rule that prescribes the dependence of the isotropic stress on the volumetric plastic strain. The yield condition (12) can also be written as

$$\sqrt{\left(\widetilde{\sigma}-\frac{\widetilde{\sigma}_c^o}{2}-\frac{\widetilde{\sigma}_c(\varepsilon_{kk}^{pl})}{2}\right)^2+\left(\frac{q}{M}\right)^2}\geqslant\frac{\widetilde{\sigma}_c^o}{2}+\frac{\widetilde{\sigma}_c(\varepsilon_{kk}^{pl})}{2} \tag{15}$$

To determine the incremental plastic strains, we specify an associated flow rule of the type

$$d\varepsilon_{ij}^{pl}=d\lambda\frac{\partial G}{\partial\widetilde{\sigma}_{ij}} \quad ; \quad G=\sqrt{\left(\widetilde{\sigma}-\frac{\widetilde{\sigma}_c^0}{2}-\frac{\widetilde{\sigma}_c(\varepsilon_{kk}^{pl})}{2}\right)^2+\left(\frac{q}{M}\right)^2}-\frac{\widetilde{\sigma}_c^0}{2}-\frac{\widetilde{\sigma}_c(\varepsilon_{kk}^{pl})}{2} \tag{16}$$

where the hardening rule takes the form

$$\widetilde{\sigma}_c=\widetilde{\sigma}_c(\varepsilon_{kk}^{pl})=H(-\varepsilon_{kk}^{pl}) \tag{17}$$

where H is a positive constant. For completeness, we note that the plastic multiplier in (16) is given by

$$\frac{d\lambda=-\frac{\partial G}{\partial\widetilde{\sigma}_{ij}}d\widetilde{\sigma}_{ij}}{\frac{\partial G}{\partial\varepsilon_{kk}^{pl}}\frac{\partial G}{\partial\widetilde{\sigma}_{ij}}\delta_{ij}=\frac{\frac{\partial G}{\partial\widetilde{\sigma}_{ij}}D_{ijkl}d\varepsilon_{kl}}{\frac{\partial G}{\partial\varepsilon_{kk}^{pl}}\frac{\partial G}{\partial\widetilde{\sigma}_{ij}}\delta_{ij}+\frac{\partial G}{\partial\widetilde{\sigma}_{ij}}}D_{ijkl}\frac{\partial G}{\partial\widetilde{\sigma}_{kl}}} \tag{18}$$

and D_{ijkl} is the elasticity tensor for the skeletal material given by

$$D_{ijkl}=\left(K_D-\frac{2}{3}G_D\right)\delta_{ij}\delta_{kl}+G_D(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}) \tag{19}$$

where K_D and G_D are, respectively, the skeletal values of the bulk modulus and the shear modulus. The incremental plastic strains are given by

$$\frac{d\varepsilon_{mn}^{pl}=-\frac{\partial G}{\partial\widetilde{\sigma}_{mn}}\frac{\partial G}{\partial\widetilde{\sigma}_{ij}}}{\frac{\frac{\partial G}{\partial\varepsilon_{kk}^{pl}}\frac{\partial G}{\partial\widetilde{\sigma}_{ij}}\delta_{ij}\,\widetilde{d\sigma_{ij}}=\frac{\partial G}{\partial\widetilde{\sigma}_{mn}}\frac{\partial G}{\partial\widetilde{\sigma}_{ij}}D_{ijkl}d\varepsilon_{kl}}{\frac{\partial G}{\partial\varepsilon_{kk}^{pl}}\frac{\partial G}{\partial\widetilde{\sigma}_{ij}}\delta_{ij}+\frac{\partial G}{\partial\widetilde{\sigma}_{ij}}D_{ijkl}\frac{\partial G}{\partial\widetilde{\sigma}_{kl}}}} \tag{20}$$

and the elastic-plastic constitutive tensor is defined by

$$d\sigma_{ij}=D_{ijkl}^{EP}d\varepsilon_{kl} \tag{21}$$

and can be expressed as

$$\frac{D_{ijkl}^{EP}=D_{ijkl}-D_{ijrs}\frac{\partial G}{\partial\widetilde{\sigma}_{rs}}\frac{\partial G}{\partial\widetilde{\sigma}_{mn}}D_{mnkl}}{\frac{\partial G}{\partial\varepsilon_{kk}^{pl}}\frac{\partial G}{\partial\widetilde{\sigma}_{ij}}\delta_{ij}+\frac{\partial G}{\partial\widetilde{\sigma}_{ij}}D_{ijkl}\frac{\partial G}{\partial\widetilde{\sigma}_{kl}}} \tag{22}$$

It must be remarked that the choices of elasto-plastic models for an exercise of this nature are many and varied. Soils possess complex non-linear constitutive relationships that can be characterized by failure, hardening and plastic flow rules of the non-associative type that can account for dilatancy; these skeletal mechanical responses can have a significant influence on post-yield pore pressure generation. Accounts of these developments can be found in the articles cited previously.

In addition to the elasto-plastic constitutive response for the porous skeleton, we assume that the fluid flow through the porous skeleton remains unchanged during yield and subsequent hardening of the porous skeleton. It is recognized that the internal damage during development of elasto-plastic effects, defect generation and void closure in consolidating and deforming media can contribute to alterations in the permeability of geomaterials (Selvadurai, 2004; Selvadurai and Głowacki, 2008, 2017).

4. Poroelastic and poroelasto-plastic response of a one-dimensional column – Numerical results

We consider the problem of a one-dimensional fluid-saturated column that is saturated with an incompressible fluid and where the porous skeleton can possess either an elastic response characterized by Hooke's law or an elasto-plastic constitutive response characterized by a soil plasticity model similar to the modified Cam Clay model with an associated flow rule and an isotropic hardening rule as described previously. The flow properties of the porous medium are defined by Darcy's law with a constant hydraulic conductivity. The constitutive properties characterizing these models are given below.

4.1. Poro-elastic Model:

Skeletal Bulk Modulus = 25 GPa; Skeletal Poisson's ratio = 0.30; Permeability of the porous skeleton = 9.85×10^{-18} m^2; Incompressible solid phase $K_S\rightarrow\infty$; nearly incompressible pore fluid.

4.2. Modified Cam Clay Poroelasto-plastic Model:

The material parameters applicable to the poroelastic model, are also applicable to the poroelasto-plastic model. In addition, the skeletal initial yield stress = 0.3 MPa and the elastic-plastic tangent bulk modulus = 10.1 MPa. The isotropic hardening rule described by (17) is selected such that$H=10^{-9}$ Pa

The material parameters chosen are only intended to serve as plausible material models for the elastic and elastoplastic responses. This range of material parameters has been used in previous poroelasto-plastic investigations (Selvadurai and Suvorov, 2012, 2014). The task of developing an analytical approach for the problems discussed here is not feasible and in the interest of the basic objectives of the paper, it is prudent to use a computational approach, utilizing the ABAQUS™ finite element code. Both the ABAQUS™ finite element code and the COMSOL™ Multiphysics code have been extensively used for modelling

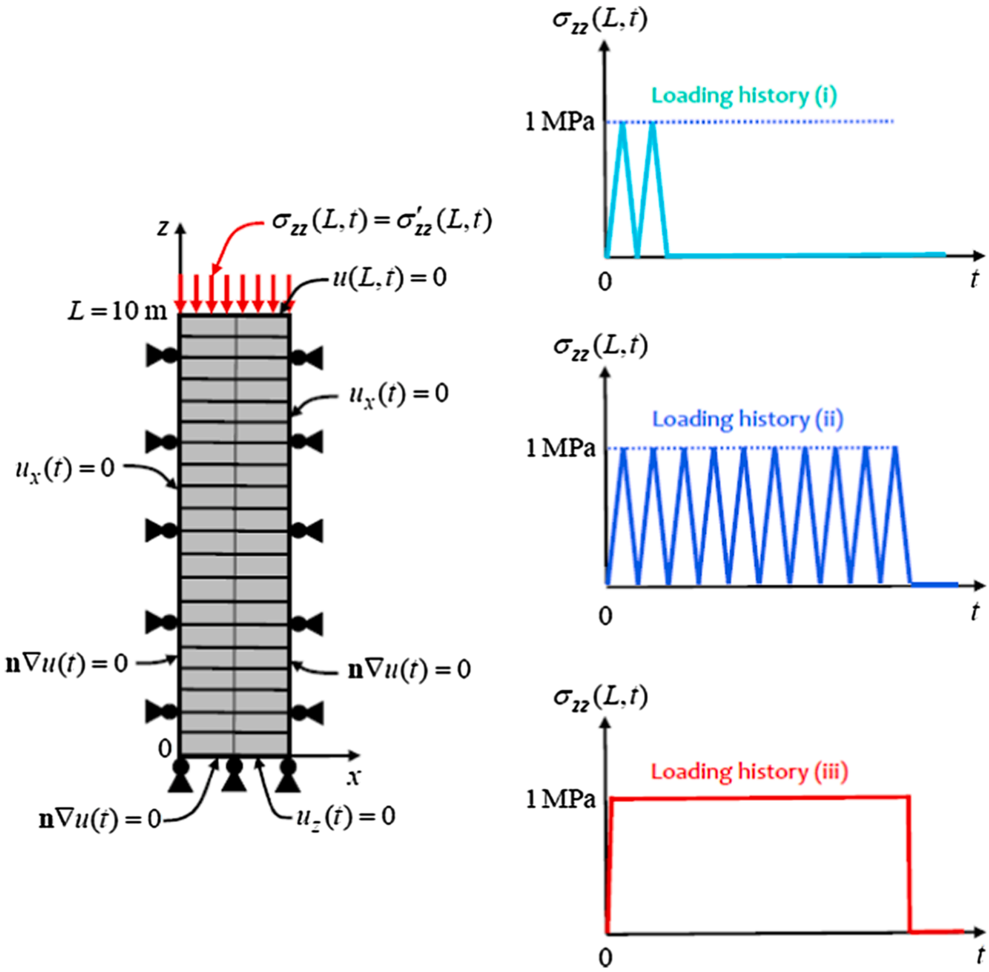

Fig. 10. The one-dimensional problem for a consolidation element and the surface loadings.

Thermal (T), Hydraulic (H) and Mechanical (M) processes in geomaterials and to investigate coupled processes involving hydro-mechanical (HM) and thermo-hydro-mechanical (THM) problems. The constitutive models are implemented in a general purpose computational multi-physics code and the initial boundary value problems are analyzed via a finite element technique with finite difference-based time integration scheme. The accuracy of the code is also verified by comparing the results obtained by the code for elastic, poroelastic and elasto-plastic problems. Several inter-code calibration exercises and comparisons with known analytical solutions have also been performed and reported in open literature. It should also be noted that the choice of the finite element modelling of the problem is largely dictated by in-house capabilities, although the use of finite-difference techniques could also be actively investigated. One-dimensional problems involving both linear and non-linear partial differential equations have been successfully treated using finite difference techniques. The application of such techniques to the class of problems that involve multiphysics, time-dependent diffusive processes and evolving material properties due to the development of failure and plastic flow merits further investigation. From a fundamental mathematical perspective, issues related to the existence of solutions, convergence, stability, uniqueness need to be addressed in relation to the non-linear problem before finite difference techniques can be employed with confidence (Lapidus and Pinder, 1999; Tadmor, 2012; Polyanin and Zaitsev, 2012; Bartels, 2015; Baines and Sarahs, 2018).

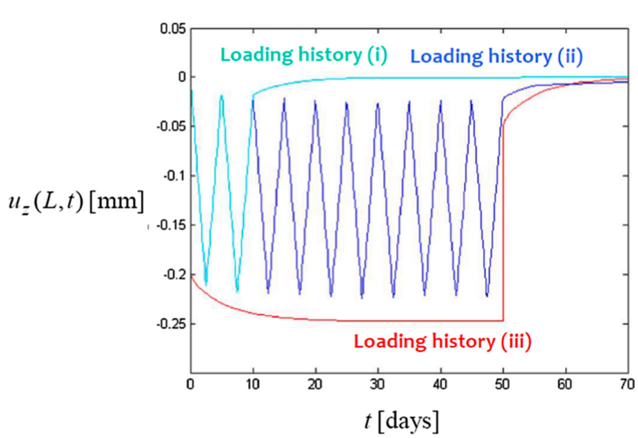

Fig. 11. Time-dependent surface displacement of the one-dimensional element-Biot poroelasticity model.

The surface of the one-dimensional column (Fig. 10) is subjected to a quasi-static normal traction with a triangular form, while maintaining zero pore pressure. The remaining lateral surfaces and the base region of the one-dimensional element are subjected to the conventional boundary conditions necessary and sufficient to create one-dimensional conditions: i.e. (i) zero shear tractions on the lateral surfaces and the base of the one-dimensional element, (ii) zero normal displacements on the lateral surfaces and the base of the element, (iii) null Neumann boundary conditions for the pore fluid pressure along the lateral surfaces and the base and (iv) zero pore fluid pressures at the surface of the element. The kinematic boundary condition at the base of the column is specified as normal displacement constrained to ensure that the rigid

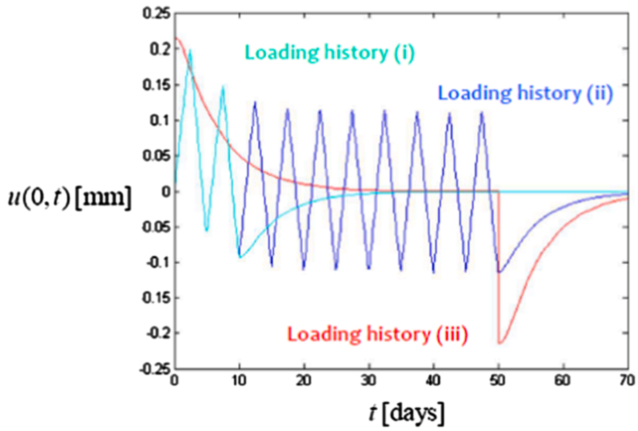

Fig. 12. Pore pressure development at the base of the one-dimensional element-Biot poroelasticity model.

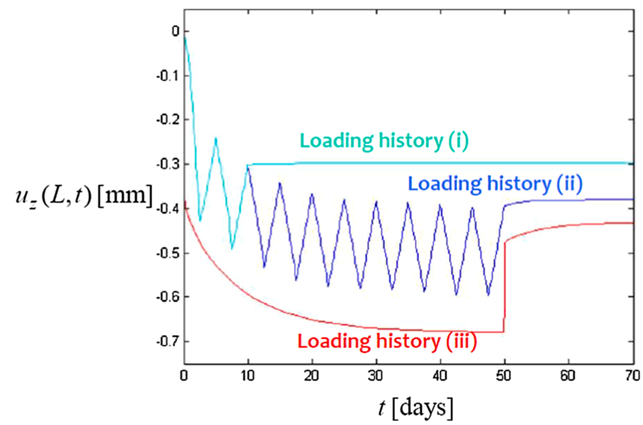

Fig. 14. The time-dependent surface displacement of the one-dimensional element-Cam Clay poroelasto-plastic model.

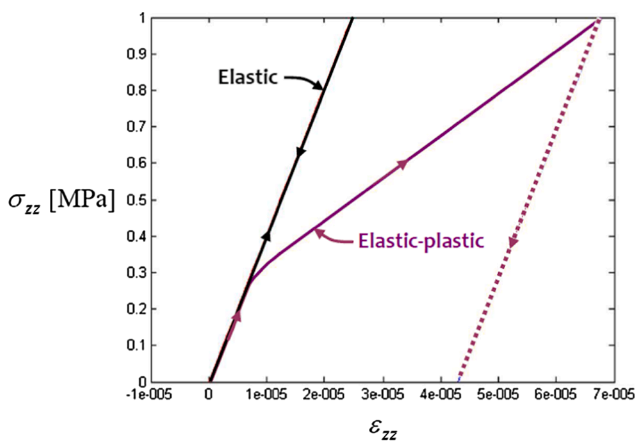

Fig. 13. The one-dimensional skeletal stress strain behaviour of the Cam Clay elasto-plastic model.

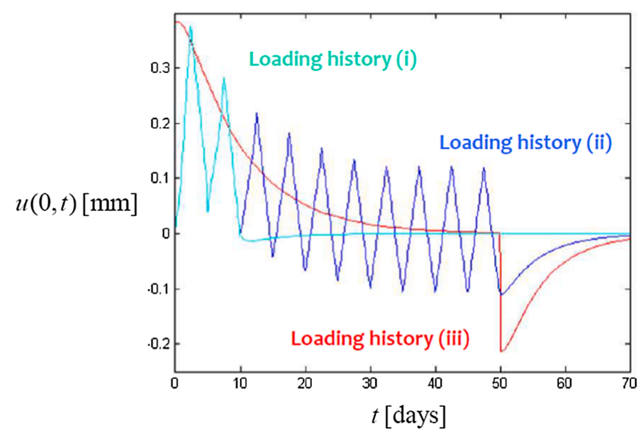

Fig. 15. Pore pressure development at the base of the one-dimensional element-Cam Clay poroelasto-plastic model.

body displacements of the modelled region are controlled. The traction boundary condition at the base will identically match the externally applied tractions, $\sigma_{zz}(L,t) = \sigma'_{zz}(L,t)$ for $\forall t \geqslant 0$. The initial conditions for the effective stress, the pore water pressure and the displacement are assumed to be zero. The upper surface of the one-dimensional element is subjected to an effective stress that varies as quasi-static loading-unloading sequences involving either two or ten loading-unloading cycles. Each triangular loading cycle lasts approximately 5 days. The third loading history involves the application of the effective stress as a near Heaviside step function applied over a finite time interval. These loading histories are illustrated in Fig. 10.

We first consider the response of the Biot poroelastic model (identical to the Terzaghi model for defining the effective stress relationship since $\alpha = 1$) for the three cases of surface loading histories. Fig. 11 illustrates the time-dependent variation in the axial surface displacement of the one-dimensional element. The displacement-time histories illustrate the type of displacements that are consistent with the loading and unloading of the element. For all three loading histories, the complete unloading of the element results in the time-dependent complete elastic recovery of the deformations of the element. Fig. 12 illustrates the time-histories of the excess pore water pressures that develop at the base of the one-dimensional element. All three loading histories contribute to the development of positive excess pore water pressures at the base of the element during the loading-unloading sequence, and upon release of the external stress, give rise to the generation of negative excess pore water pressure. In the case of the Heaviside step function form of loading, the negative excess pore fluid pressure generated at the base of the element is identical to the positive excess pore fluid pressure generated during the application of the step function form of loading, indicative of the response suggested in Fig. 9 (*i*). At the end of the loading histories, the surface displacements are fully reversed and the excess pore fluid pressures reduce to zero.

We next consider the response of the one-dimensional element where the skeletal response is defined by the Cam Clay elasto-plastic model discussed previously. Fig. 13 illustrates the axial stress vs. axial strain response for the porous skeleton. The fluid-saturated poroelasto-plastic one-dimensional element is subjected to the three loading histories shown in Fig. 10. The displacement and pore fluid pressure boundary conditions and the stresses applied to the surface of the one-dimensional element are identical to those used in the analysis of the poroelasticity problem (Fig. 10). Attention is restricted to the computational results for the axial displacement at the point of application of the one-dimensional axial stress and the pore fluid pressures observed at the base of the one-dimensional element. Fig. 14 illustrates the time-dependent variation in the surface displacement of the one-dimensional element. As is evident, all three loading histories contribute to irreversible deformations of varying magnitude in the one-dimensional element. The time-dependent evolution of deformation during loading is also distinctly different from the time-dependent recovery during the unloading phase indicative of the influence of the stiffnesses during the loading and the unloading phase on the elastic recovery. Fig. 15 shows the variation of excess pore fluid pressure at the base of the one-dimensional element with a Cam Clay-based poroelasto-plastic response. Again, negative excess pore

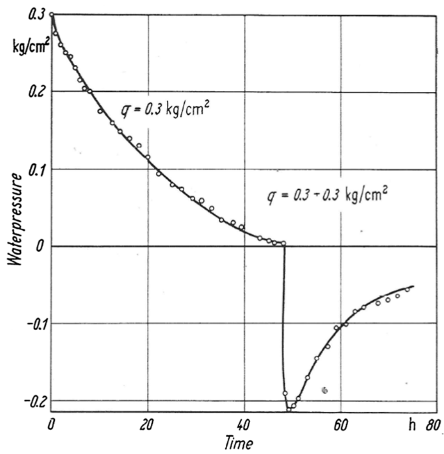

Fig. 7. Positive waterpressure during loading and negative waterpressure after unloading.

Fig. 16. Pore pressure development during loading and unloading of a clay [After Tan (1966)]

fluid pressure develops during instantaneous unloading but, unlike in the purely poroelastic case, the magnitude during unloading is different from the magnitude during loading. This phenomenon is a common observation in the testing of clay soils. The early experiments conducted by Tan (1966) clearly indicated the development of negative pore fluid pressures during unloading (Fig. 16). At the termination of the *unconsolidation* process, the negative pore pressures for all three loading histories reduce to zero.

5. Concluding remarks

The concept of recoverability of skeletal deformation is entrenched in the application of the one-dimensional theory of consolidation proposed by Terzaghi and extended to three dimensions by Biot. Under very restricted monotonic loading conditions, the classical theory of consolidation is an acceptable outcome of these basic approaches. When quasi-static loading cycles are encountered, the irreversibility of the skeletal deformations need to be accommodated. The absence of the discussion of irreversibility of skeletal deformations can lead to unacceptable interpretation of the pore pressure development within the consolidating element. The influence of irreversibility of skeletal deformations during a cycle of loading and unloading needs to be examined within the context of a computational approach. Such extensions are relatively straightforward and the inclusion of the basic findings of studies similar to those presented in this article will complete a much-needed addendum to Terzaghi's theory of consolidation and place this key theory in the development of the subject of soil mechanics in the correct setting. The paper also points to new research directions where the consolidation of geomaterials can be examined in the context of bi-modulus behaviour of the geomaterial skeleton and subject to unilateral constraints imposed by soil reinforcement.

Declaration of Competing Interest

The author declares that he has no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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References

Agbezuge, L.K., Deresiewicz, H., 1974. On the indentation of a consolidating half-space. Isr. J. Technol. 12, 322–338.
Agbezuge, L.K., Deresiewicz, H., 1975. The consolidation of a circular footing. Isr. J. Technol. 13, 264–269.
Aleynikov, S., 2011. Spatial Contact Problems in Geotechnics. Boundary Element Method. Springer-Verlag, Berlin.
Ambartsumian, S.A., 1966. Equations of the plane problem of the elastic theory of materials with different moduli in tension and compression. Proc. Acad. Sci. Armenia SSSR, Mechanics 19, 1–19.
Atkin, R.J., Craine, R.E., 1976. Continuum theories of mixtures: Applications. J. Inst. Math. Applic. 17, 153–207.
Baines, M.J., Sarahs, N., 2018. A moving mesh finite difference scheme that preserves scaling symmetry for a class of non-linear diffusion problems. J. Comp. Appl. Math 340, 380–389.
Bartels, S., 2015. Numerical Methods for Nonlinear Partial Differential Equations, Springer Series in Computational Mathematics 47. Springer-Verlag, Berlin.
Berryman, J.G., 1992. Effective stress for transport properties of inhomogeneous porous rock. J. Geophys. Res. 97, 17409–17424.
Bert, C.W., 1977. Models of fibrous composites with different properties in tension and compression. J. Eng. Mat. Technol. 99, 344–349.
Biot, M.A., 1935. Le problème de la consolidation des matères argileuses sous une charge. Ann. Soc. Scientif Bruxelles, série 55, 110–113.
Biot, M.A., 1941. General theory of three-dimensional consolidation. J. Appl. Phys. 12, 155–164.
Biot, M.A., 1956. General solutions of the equations of elasticity and consolidation for a porous material. J. Appl. Mech. 23, 91–96.
Bishop, A.W., 1973. Influence of an undrained change in stress on the pore pressures in porous media of low compressibility. Geotechnique 23, 435–442.
Borja, R.I., Tamagnini, C., 1998. Cam-Clay plasticity Part III: Extension of the infinitesimal model to include finite strains. Comp. Meth. Appl. Mech. Eng. 155, 73–95.
Borja, R.I., Tamagnini, C., Alarćon, E., 1998. Elastoplastic consolidation at finite strain. Part 2: Finite element implementation and numerical examples. Comp. Meth. Appl. Mech. Eng. 159, 103–122.
Bowen, R.M., 1976. Theory of Mixtures. In. Continuum Physics, (AC Eringen, Ed.) 3, Academic Press, San Diego., pp. 1-127.
Bowen, R.M., Wiese, J.C., 1969. Diffusion in mixtures of elastic materials. Int. J. Eng. Sci. 7, 689–722.
Braun, P., Ghabezloo, S., Delage, P., Sulem, J., Conil, N., 2019. Determination of multiple thermo-hydro-mechanical rock properties in a single transient experiment: Application to shales. Rock Mech. Rock Eng. 52, 2030–2038.
Burland, J.B., 1989. Ninth Laurits Bjerrum Memorial Lecture: "Small is Beautiful:-the stiffness of soils at small strains". Can. Geotech. J. 26, 499–516.
Carslaw, H.S., Jaeger, J.C., 1959. Heat Conduction in Solids. Oxford University Press, Oxford.
Carter, J.P., Booker, J.R., Small, J.C., 1979. The analysis of finite elasto-plastic consolidation. Int. J. Num. Analyt. Meth. Geomech. 3, 107–129.
Chau, K.T., 2013. Analytic Methods in Geomechanics. CRC Press, Boca Raton FL.
Chen, Y., Selvadurai, A.P.S., Liang, W., 2018. Computational modelling of groundwater inflow during a long wall coal mining advance: A case study from the Shanxi Province. China. Rock Mech. Rock Eng. 52, 917–934.
Chen, W.F., Baladi, G.Y., 1985. Soil Plasticity: Theory and Implementation, Developments in Geotechnical Engineering 38. Elsevier Scientific publishing, Amsterdam.
Cheng, A.H.-D., 2015. Poroelasticity. Springer-Verlag, Berlin.
Chiarella, C., Booker, J.R., 1975. The time-settlement behaviour of a rigid die resting on a deep clay layer. Quart. J. Mech. Appl. Math. 28, 317–328.
Coussy, O., 1995. Mechanics of Porous Media. John Wiley, New York.
Crank, J., 1970. The Mathematics of Diffusion. Clarendon Press, Oxford.
Cryer, C.W., 1963. A comparison of the three-dimensional consolidation theories of Biot and Terzaghi. Quart. J. Mech. Appl. Math. 16, 401–412.
Darve, F. (Ed.), 1990. Geomaterials: Constitutive Equations and Modelling. Elsevier Applied Science, The Netherlands.
Davis, R.O., Selvadurai, A.P.S., 1996. Elasticity and Geomechanics. Cambridge University Press, Cambridge.
Davis, R.O., Selvadurai, A.P.S., 2005. Plasticity and Geomechanics. Cambridge University Press, Cambridge.

de Boer, R., 2000. Theory of Porous Media. Highlights in Historical Developments and Current State. Springer-Verlag, Berlin, Heidelberg.
de Boer, R., Ehlers, W., 1990. The development of the concept of effective stresses. Acta Mechanica 83, 77–92.
de Josselin de Jong, G., 1953. Consolidation around pore pressure meters. J. Appl. Phys., 24, 922–928.
de Josselin de Jong, G., 1957. Application of stress functions to consolidation problems. Proc. 4th Int. Conf. Soil Mech. Foundation Eng. 1, 320-323.
Desai, C.S., 2001. Mechanics of Materials and Interfaces: The Disturbed State Concept. CRC Press, Boca Raton, FL.
Desai, C.S., Siriwardane, H.J., 1984. Constitutive Laws for Engineering Materials with Emphasis on Geologic Materials. Prentice-Hall, Englewood Cliffs, NJ.
Detournay, E., Cheng, A.H.-D., 1993. Fundamentals of poroelasticity, Ch. 5 in Comprehensive Rock Engineering: principles, Practice and projects, Vol. II Analysis and Design methods, C. Fairhurst, Ed.. Pergamon Press, 113-171.
Drucker, D.C., Prager, W., 1952. Soil mechanics and plastic analysis or limit design. Quart. Appl. Math. 10, 157–165.
Drucker, D.C., Gibson, R.E., Henkel, D.J., 1957. Soil mechanics and work hardening theories of plasticity. Trans. Amer. Soc. Civil Eng. 122, 338–346.
Ehlers, W., Bluhm, J. (Eds.), 2002. Porous Media: Theory, Experiments and Numerical Applications, Springer-Verlag, Berlin.
Ehlers, W., 1991. Toward finite theories of liquid-saturated elasto-plastic porous media. Int. J. Plasticity 7, 433–475.
Fillunger, P., 1936. Erdbaumechanik? Selbstverlag des Verfassers, Wien.
Florin, V.A., 1948. The Theory of Consolidation of Earth Masses (In Russian). Gosstroiizdat, Leningrad.
Gens, A., 2010. Soil-environment interactions in geotechnical engineering. Géotechnique 60, 3–74.
Geuze, E.C.W.A., Tan, T.K., 1950. The shearing properties of soil. Géotechnique 2, 141–161.
Gibson, R.E., 1974. The analytical method in soil mechanics. Géotechnique 24, 115–140.
Gibson, R.E., Lo, K.Y., 1961. A theory of consolidation for soils exhibiting secondary compression. Norwegian Geotechnical Institute Publication No. 41, Acta Polytechnica Scandinavica, 296/191, Ci 10.
Gibson, R.E., McNamee, J., 1963. A three-dimensional problem of the consolidation of a semi-infinite clay stratum. Quart. J. Mech. Appl. Math. 16, 115–127.
Gibson, R.E., England, G.L., Hussey, M.J.L., 1967. The theory of one-dimensional consolidation of saturated clays. I. Finite non-linear consolidation of thin homogeneous layers. Géotechnique 17, 261–273.
Gibson, R.E., Schiffman, R.L., Cargill, K.W., 1981. The theory of one-dimensional consolidation of saturated clays. II Finite non-linear consolidation of thick homogeneous layers. Can. Geotech. J. 18, 280–293.
Gibson, R.E., Gobert, A., Schiffman, R.L., 1989. On Cryer's problem with large displacements. Int. J. Numer. Analyt. Methods Geomech. 13, 251–262.
Gibson, R.E., Knight, K., Taylor, P.W., 1963. A critical experiment to examine theories of three-dimensional consolidation. Proc. European Conf. Soil Mech. Foundation Eng., Wiesbaden, 69–76.
Green, A.E., Adkins, J.E., 1970. Large Elastic Deformations. Clarendon Press, Oxford.
Green, A.E., Mkrtichian, J.Z., 1977. Elastic solids with different moduli in tension and compression. J. Elasticity 7, 369–386.
Green, A.E., Naghdi, P.M., 1965. A dynamical theory of interacting continua. Int. J. Eng. Sci. 3, 231–241.
Green, A.E., Naghdi, P.M., 1967. Remarks on a paper by RM Bowen. Arch. Rational Mech. Anal. 27, 175–180.
Harr, M.E., 1966. Foundations of Theoretical Soil Mechanics. McGraw-Hill Book Co, New York.
Hill, J.M., Dewynne, J., 1987. Heat Conduction. Blackwell Scientific Publications, Oxford.
Ichikawa, Y., Selvadurai, A.P.S., 2012. Transport Phenomena in Porous Media: Micro/ Macro Behaviour. Springer Verlag, Berlin.
Jones, R.M., 1977. Stress-strain relations for materials with different moduli in tension and compression. Am. Inst. Aeronaut. Astronaut. J. 15, 16–23.
Kelvin, Lord (W. Thompson), 1878. On the thermo-elastic, thermo-magnetic and pyroelectric properties of matter, Quart. J. Math., 1, 4–27.
Kravtchenko, J., Sirieys, P.M. (Eds), 1966. Rheology and Soil Mechanics. Proceedings of the IUTAM Conference, Grenoble, 1964, Springer-Verlag, Berlin.
Krynine, D.P., 1947. Soil Mechanics. Its Principles and Structural Applications. McGraw-Hill Book Co Inc., New York.
Lamb Jr, G.L., 1995. Introductory Applications of Partial Differential Equations. With Emphasis on Wave Propagation and Diffusion. John Wiley and Sons Inc., New York.
Lan, Q., Selvadurai, A.P.S., 1996. Interacting indenters on a poroelastic half-space. J. Appl. Math. Phys. (ZAMP) 47, 695–716.
Lapidus, L., Pinder, G.F., 1999. Numerical Solution of Partial Differential Equations in Science and Engineering. Wiley-Interscience, New York.
Lee, E.H., 1969. Elastic-plastic deformation at finite strains. J. Appl. Mech Trans. ASME 36, 1–6.
Lewis, R.W., Schrefler, B.A., 1998. The Finite Element Method in the Static and Dynamic Deformation and Consolidation of Porous Media. John Wiley, New York.
Lubarda, V.A., 2004. Constitutive theories based on multiplicative decomposition of deformation gradient: Thermoelasticity, elastoplasticity and biomechanics. Appl. Mech. Rev. 57, 95–108.
Mandel, J., 1950. Etude mathématiques de la consolidation des sols. Actes du colloque International de mécanique, Poitiers 4, 9–19.
Mandel, J., 1953. Consolidation des sols (Etude mathématiques). Géotechnique 7, 287–299.
Mandel, J., 1957. Consolidation des Couches ď Argiles. Proc. 4th Int. Conf. Soil Mech. Foundation Eng., London 1, 360-367.
Mavko, G., Mukerji, T., Dvorkin, J., 2009. The Rock Physics Handbook: Tools for Seismic Analysis of Porous Media. Cambridge University Press, Cambridge.
McNamee, J., Gibson, R.E., 1960a. Displacement functions and linear transforms applied to diffusion through porous elastic media. Quart. J. Mech. Appl. Math. 13, 98–111.
McNamee, J., Gibson, R.E., 1960b. Plane strain and axially symmetric problems of the consolidation of a semi-infinite clay stratum. Quart. J. Mech. Appl. Math. 13, 210–227.
Mikasa, M., 1965. The consolidation of soft clay. Jap. Soc. Civil Engs. 21, 21–26.
Mills, N., 1966. Incompressible mixtures of Newtonian fluids. Int. J. Eng. Sci. 4, 97–112.
Mkrtichian, R.E., 1970. On a model of a medium hetero-resistant to tension and compression. Proc. Acad. Sci Armenian SSSR, Mech. 23, 37–47.
Pande, G.N., Pietruszczak, S., 1990. A rational interpretation of pore pressure parameters. Geotechnique 40, 275–279.
Pande, G.N., Beer, G., Williams, J.R., 1990. Numerical Methods in Rock Mechanics. John Wiley, Chichester, UK.
Paria, G., 1963. Flow of fluids through porous deformable solids. Appl. Mech. Rev. 16, 901–907.
Pariseau, W.G., 1999. Poroelastic-plastic consolidation - Analytical solution. Int. J. Numer. Analyt. Methods Geomech. 23, 577–594.
Pietruszczak, S., 2010. Fundamentals of Plasticity in Geomechanics. CRC Press, Boca Raton, FL.
Polyanin, A.D., Zaitsev, V.F., 2012. Handbook of Nonlinear Partial Differential Equations, 2nd Ed. CRC Press, Boca Raton, FL.
Puzrin, A.M., 2012. Constitutive Modelling in Geomechanics. Springer-Verlag, Berlin.
Rajapakse, R.K.N.D., Senjuntichai, T., 1993. Fundamental solutions for a poroelastic halfspace with compressible constituents. J. Appl. Mech. Trans. ASME 60, 847–856.
Rendulic, L., 1936. Porenziffer und Porenwasserdruck in Tonen. Der Bauingenieur 17, 559–564.
Rice, J.R., 1992. Fault states, pore pressure distribution and the weakness of the San Andreas fault. In: Evans, B. and T-F Wong, T.-F. (Eds.), Fault Mechanics and Transport Properties of Rocks, Academic Press, San Diego, Ch. 20, pp. 475–503.
Rice, J.R., Cleary, M.P., 1976. Some basic stress diffusion solutions for fluid-saturated elastic porous media with compressible constituents. Rev. Geophys. Space Phys. 14, 227–241.
Rice, J.R., Rudnicki, J.W., Simons, D.A., 1978. Deformation of spherical cavities and inclusions in fluid-infiltrated elastic materials. Int. J. Solids Struct. 14, 289–303.
Rivlin, R.S., 1953. The solution of problems in second-order elasticity theory. J. Ration. Mech. Anal. 2, 53–81.
Rivlin, R.S., 1961. Some topics in finite elasticity. In: Goodier, J.N. and Hoff, N.J. (Eds), Structural Mechanics, Proc. First Symposium on Naval Structural Mechanics, Palo Alto, CA, Pergamon Press, Oxford, pp. 169-198.
Schiffman, R.L., Ladd, C.C., Chen, A.T.-F., 1966. The secondary consolidation of clay, In: Kravtchenko, J. and Sirieys, P.M. (Eds), Rheology and Soil Mechanics, Proc. IUTAM Symposium, Grenoble, Springer-Verlag, Berlin, pp. 256-270.
Schiffman, R.L., 1980. Finite and infinitesimal strain consolidation. J. Geotech. Eng. Div ASCE 106, 115–119.
Schiffman, R.L., 1984. A bibliography on soil consolidation. In: Bear, J., Corapcioglu, M. Y. (Eds.), Fundamentals of Transport in Porous Media. Martinus Nijhoff Publishers, Dordrecht, The Netherlands, pp. 617–669.
Schiffman, R.L., Cargill, K.W., 1981. Finite strain consolidation of sedimenting clay deposits, Proc.10th Int. Conf. Soil Mech. Foundation Engng, Stockholm, 1, 239–242.
Schofield, A.N., Wroth, C.P., 1968. Critical State Soil Mechanics. McGraw-Hill, London.
Schrefler, B.A., 2002. Mechanics and thermodynamics of saturated/unsaturated porous materials and quantitative solutions. Appl. Mech. Rev. 55, 351–388.
Schwartz, R.T., Schwartz, H.S., 1967. Characteristics of boron fibers and boron-fiber-reinforced elastic composites. Amer. Inst. Aeronaut. Astronaut. J. 5, 289–295.
Scott, R.F., 1963. Principles of Soil Mechanics. Addison-Wesley Publ. Co., Inc., Reading, MA.
Selvadurai, A.P.S., 1979. Elastic Analysis of Soil-Foundation Interaction, Developments in Geotechnical Engineering, Vol 17. Elsevier Scientific Publishing, Amsterdam.
Selvadurai, A.P.S. (Ed.), 1996. Mechanics of Poroelastic Media. Kluwer Academic Publishers, The Netherlands.
Selvadurai, A.P.S., 2000. Partial Differential Equations in Mechanics. Fundamentals, Laplace's Equation, the Diffusion Equation, the Wave Equation Vol. 1.
Selvadurai, A.P.S., 2004. Stationary damage modelling of poroelastic contact. Int. J. Solids Struct. 41, 2043–2064.
Selvadurai, A.P.S., 2007. The analytical method in geomechanics. Appl. Mech. Rev. 60, 87–106.
Selvadurai, A.P.S., 2019. The Biot coefficient for a low permeability heterogeneous limestone. Cont. Mech. Thermodyn. 31, 939–953.
Selvadurai, A.P.S., 2020. In-plane loading of a bonded rigid disc inclusion embedded at a pre-compressed elastic interface: The role of non-linear interface responses. Mech. Systs. Signal Processing 144, 106871.
Selvadurai, A.P.S., Atluri, S.N. (Eds.), 2010. Contact Mechanics in the Engineering Sciences. Material Characterization, Micromechanical Processes and Modelling in Geosciences, Science Tech, GA.
Selvadurai, A.P.S., Boulon, M.J. (Eds), 1995. Mechanics of Geomaterial Interfaces. Studies in Applied Mechanics, 42, Elsevier Sciences BV, The Netherlands.
Selvadurai, A.P.S., Głowacki, A., 2008. Evolution of permeability hysteresis of Indiana Limestone during isotropic compression. Ground Water 46, 113–119.
Selvadurai, A.P.S., Głowacki, A., 2017. Stress-induced permeability alterations in an argillaceous limestone. Rock Mech. Rock Eng. 50, 1079–1096.

Selvadurai, A.P.S., Kim, J., 2016. Poromechanical behaviour of a surficial geological barrier during fluid injection into an underlying poroelastic storage formation. Proc. Roy. Soc., Math. Phys. Sci., Series A, doi.org/10.1098/rspa.2015.0418.
Selvadurai, A.P.S., Mahyari, A.T., 1998. A cavity expansion problem for a bi-modulus poroelastic medium. In: Thimus, J.-F., Abousleiman, Y., Cheng, A.H.-D., Coussy, O. and Detournay, E. (Eds). Proc. Symposium: Poromechanics, A Tribute to Maurice A. Biot, Liege, Belgium. A.A. Balkema, The Netherlands, 499-504.
Selvadurai, A.P.S., Najari, M., 2017. Thermo-hydro-mechanical behaviour of the argillaceous Cobourg Limestone. J. Geophys. Res. (Solid Earth) 122, 4157–4171.
Selvadurai, A.P.S., Samea, P., 2021. Mechanics of a pressurized penny-shaped crack in a poroelastic halfspace. Int. J. Eng. Sci. https://doi.org/10.1016/j.ijengsci.2021.103472.
Selvadurai, A.P.S., Shirazi, A., 2004. Mandel-Cryer effects in fluid inclusions in damage susceptible poroelastic geologic media. Comp. Geotech. 31, 285–300.
Selvadurai, A.P.S., Spencer, A.J.M., 1972. Second–order elasticity with axial symmetry, I-General Theory. Int. J. Eng. Sci. 10, 97–114.
Selvadurai, A.P.S., Suvorov, A.P., 2012. Boundary heating of poroelastic and poro-elastoplastic spheres. Proc. Roy. Soc., Math. Phys. Sci. Series A 468, 2779–2806. https://doi.org/10.1098/rspa.2012.0035.
Selvadurai, A.P.S., Suvorov, A.P., 2014. Thermo-poromechanics of a fluid-filled cavity in a fluid-saturated porous geomaterial. Proc. Roy. Soc., Math. Phys. Sci. Series A, 470, Article No. 2163; pp 1-23, (2163): https://doi.org/10.1098/rspa.2013.0634.
Selvadurai, A.P.S., Suvorov, A.P., 2016a. Thermo-poroelasticity and Geomechanics. Cambridge University Press, Cambridge.
Selvadurai, A.P.S., Suvorov, A.P., 2016b. Coupled hydro-mechanical effects in a poro-hyperelastic material. J. Mech. Phys. Solids 91, 311–333.
Selvadurai, A.P.S., Suvorov, A.P., 2017. On the inflation of poro-hyperelastic annuli. J. Mech. Phys. Solids 107, 229–252.
Selvadurai, A.P.S., Suvorov, A.P., 2018. On the development of instabilities in an annulus and a shell composed of a poro-hyperelastic material. Proc. Roy. Soc., Math. Phys. Sci. Series A, 474, Art. ID 20180239; https://dx.doi.org/10.6084/m9.figshare.c.4271114.
Selvadurai, A.P.S., Suvorov, A.P., 2020. Influence of pore shape on the bulk modulus and the Biot coefficient of fluid-saturated porous rock. Scientific Reports 10, 18959. https://doi.org/10.1038/s41598-020-75979-6.
Selvadurai, A.P.S., Yu, Q., 2006. On the indentation of a polymeric membrane. Proc. Roy. Soc. Math. Phys. Sci. Series A 462, 189–209.
Selvadurai, A.P.S., Selvadurai, P.A., Nejati, M., 2019. A multiphasic approach for estimating the Biot coefficient for Grimsel granite. Solid Earth 10, 2001–2014.
Skempton, A.W., 1954. The pore pressure coefficients A and B. Geotechnique 4, 143–147.
Skempton, A.W., 1960. Significance of Terzaghi's concept of effective stress (Terzaghi's discovery of effective stress). In: Bjerrum, L., Casagrande, A., Peck, R.B., Skempton, A.W. (Eds.), From Theory to Practice in Soil Mechanics. John Wiley and Sons, London, pp. 45–53.
Small, J.C., Booker, J.R., Davis, E.H., 1976. Elasto-plastic consolidation of soil. Int. J. Solids Struct. 12, 431–448.
Souley, M., Lopez, P., Boulon, M.J., Thoraval, A., 2015. Experimental hydromechanical characterization and numerical modelling of a fractured and porous sandstone. Rock Mech. Rock Eng. 48, 1143–1161.
Spence, D.A., Mkrtchian, J.Z., 1977. The Boussinesq problem for a material with different moduli in tension and compression. Quart. J. Mech. Appl. Math. 30, 449–466.
Spencer, A.J.M., 1970. The static theory of finite elasticity. J. Inst. Math. Appl. 6, 164–200.
Stickle, M.M., Pastor, M., 2018. A practical analytical solution for one-dimensional consolidations. Géotechnique 68, 786–793.
Suklje, L., 1969. Rheological Aspects of Soil Mechanics. Wiley, Interscience, New York.
Suvorov, A.P., Selvadurai, A.P.S., 2019. The Biot coefficient for an elasto-plastic material. Int. J. Eng. Sci. 145, 103166 https://doi.org/10.1016/j.ijengsci.2019.103166.
Suvorov, A.P., Selvadurai, A.P.S., 2016. On poro-hyperelastic shear. J. Mech. Phys. Solids 96, 445–459.
Tadmor, E., 2012. A review of numerical methods for nonlinear partial differential equations. Am. Math. Soc. 49, 507–554.
Tan, T.K., 1966. Determination of the rheological parameters and the hardening coefficients of clays. In: Kravtchenko, J., Sirieys, P.M. (Eds), Rheology and Soil Mechanics, Proc. IUTAM Symposium, Grenoble, Springer-Verlag, Berlin, 256–270.
Taylor, D.W., 1948. Fundamentals of Soil Mechanics. John Wiley, New York.
Taylor, D.W., Merchant, W., 1940. A theory of clay consolidation accounting for secondary compression. J. Math. Phys. 19, 167–185.
Terzaghi, K., 1923. Erdbaumechanik. Franz Deuticke, Vienna.
Terzaghi, K., 1924. Die Theorie der hydrodynamischen Spannungserscheinungen ind ihr erdbautechnisches Anwendungsgebeit. Proc. First Int. Congress of Applied Mechanics, (CB Bizeno, JM Burgers Eds).
Terzaghi, K., 1927. Principles of final soil classification. Public Roads 8, 41–53.
Terzaghi, K., 1936. The shearing resistance of saturated soils and the angle between the planes of shear. Proc. First Int. Conf. Soil Mech. Foundation Engineering, Harvard University, 1, 54-56.
Terzaghi, K., 1943. Theoretical Soil Mechanics. John Wiley and Sons Inc., New York.
Terzaghi, K., Fröhlich, O.K., 1936. Theorie der Setzung von Tonschichten. Ein Einführung in die analytische Tonmechanik, Franz Deuticke, Leipzig un Wein.
Terzaghi, K., Peck, R.B., 1948. Soil Mechanics in Engineering Practice. John Wiley and Sons Inc, New York.
Timoshenko, S.P., 1930. Strength of Materials, Part I. Elementary Theory and Problems. Part II. Advanced Theory and Problems, D van Nostrand Co Inc, New York.
Treloar, L.R.G., 1976. The mechanics of rubber elasticity. Proc. Roy. Soc. Series A Math. Phys. Sci. A351, 301–330.
Tschebotarioff, G.P., 1951. Soil Mechanics, Foundations and Earth Structures. McGraw-Hill Book Co Inc., New York.
Verruijt, A., 2015. Theory and Problems of Poroelasticity. Delft University of Technology.
Voigt, W., 1910. Lehrbuch der Krystallphysik. BG Teubner, Berlin.
Wan, R., Nicot, F., Darve, F., 2016. Failure in Geomaterials. A Contemporary Treatise. ISTE Press, Elsevier.
Wang, H.F., 2000. Theory of Linear Poroelasticity with Applications to Geomechanics and Hydrology. Princeton University Press, NJ.
Willner, K., 2003. Kontinuums-und Kontaktmechanik. Synthetische und analytische Darstellung. Springer-Verlag, Berlin.
Yue, Z.Q., Selvadurai, A.P.S., 1995. On the mechanics of a rigid disc inclusion embedded in a fluid saturated poroelastic medium. Int. J. Eng. Sci. 33, 1633–1662.
Zaretskii, Yu. K., 1972. Theory of Soil Consolidation, Israel Program for Scientific Translations, Jerusalem.
Zaslavsky, D., 1964. Saturated and unsaturated flow equation in an unstable porous medium. Soil Sci. 98, 317–321.
Zhang, D., Kang, Y., Selvadurai, A.P.S., You, L., 2019. Experimental investigation of the effect of salt precipitation on the physical and mechanical properties of a tight sandstone. Rock Mech. Rock Eng., doi.org/10.1007/s00603-019-02032-y.