THE DISTRIBUTION OF STRESS IN A RUBBER-LIKE ELASTIC MATERIAL BOUNDED INTERNALLY BY A RIGID SPHERICAL INCLUSION SUBJECTED TO A CENTRAL FORCE

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(Received 4 November 1974; accepted as ready for print 4 April 1975)

Introduction

In this paper exact closed form solutions are obtained for the second-order effects in an infinite incompressible elastic medium containing a bonded rigid spherical inclusion which is subjected to a central force. The second-order elasticity theory adequately describes the mechanical behaviour of most rubber-like materials at moderately large strains. A displacement function technique is employed in the solution of the second-order problem.

Analysis

The differential equations which arise in formulating the mathematical theory of highly elastic rubber-like materials are generally non-linear in character. Approximate methods of analysis are therefore of particular value in instances where exact solutions of these non-linear differential equations are not readily obtainable. The method of successive approximations is one such technique which has received considerable interest. Second-order elasticity theory considers the successive approximation procedure to include terms which are quadratic in the displacement gradients, and adequately describes the mechanical behaviour of most rubber-like materials at moderately large strains.

The general theory of second-order elasticity for axially symmetric deformations of isotropic incompressible elastic...
materials was developed by Selvadurai and Spencer [1]. This particular method of analysis of second-order effects is facilitated by the introduction of a displacement function, $\Psi$, which reduces the problem to the solution of a single equation of the form $E^2 \Psi = f(S, \theta)$, where $E^2$ is Stokes' differential operator, $f(S, \theta)$ depends only on the first-order or classical elasticity solution and $S, \theta$ are the spherical polar coordinates of the reference configuration.

In this paper we consider the problem of an incompressible infinite elastic medium containing a bonded rigid spherical inclusion which is subjected to a central resultant force. This problem is of importance in the analysis of stress concentrations in multiphase rubber-like materials and in bonded rubber mountings. It also serves as a useful mechanical analogue of Kelvin's problem for the concentrated force acting at the interior of an infinite elastic solid. The solutions for the second-order displacement and stress components are presented in exact closed form.

Basic equations

The general theory of second-order elasticity is given in Green and Adkins [2]. A detailed account of the displacement function formulation of the axially symmetric second-order problem is given by Selvadurai and Spencer [1]. For completeness, we shall briefly outline the relevant results.

The spherical polar coordinates of the reference configuration are denoted by $(S, \phi, \theta)$ such that

$$R = S \sin \theta, \quad Z = S \cos \theta, \quad (1)$$

where $R, Z$ are the cylindrical polar coordinates. We restrict our attention to a state of stress which is symmetric about the $Z$-axis. By using the displacement function technique the solution of the first-order or the classical elasticity problem is reduced to the solution of the equations

$$E^2 \Psi_1 = 0, \quad \nabla^2 p_1 = 0, \quad (2)$$

where $\Psi_1$ is the first-order displacement function and $p_1$ is the first-order hydrostatic pressure, subject to the particular
boundary conditions of the problem. The operators $E^2$ and $V^2$ are, respectively, Stokes' and Laplace's differential operators given by

$$E^2 = \frac{\partial^2}{\partial S^2} + \frac{1}{S^2} \frac{\partial^2}{\partial \theta^2} - \cot \theta \frac{\partial}{\partial \theta}, \quad (3)$$

$$V^2 = \frac{\partial^2}{\partial S^2} + \frac{2}{S} \frac{\partial}{\partial S} + \frac{1}{S^2} \frac{\partial^2}{\partial \theta^2} + \cot \theta \frac{\partial}{\partial \theta}, \quad (4)$$

and

$$E^4 = E^2 E^2.$$

The first-order displacement components $u_1$ and $w_1$ in the $S, \theta$ directions are completely determined from the first-order displacement function $\psi_1$ by means of the relations

$$u_1 = -\frac{1}{S^2 \sin \theta} \frac{\partial \psi_1}{\partial \Phi}, \quad w_1 = \frac{1}{S \sin \theta} \frac{\partial \psi_1}{\partial \Phi}. \quad (5)$$

The first-order constitutive equation is

$$T_1 = -p_1 I + \mu C_1, \quad (6)$$

where $T_1$ is the first-order component of the Cauchy stress tensor; $\mu$ is the linear elastic shear modulus; $p_1$ is the first-order hydrostatic pressure; $I$ is the unit matrix and

$$C_1 = G_1 + G_1^T. \quad (7)$$

where

$$G_1 = \begin{bmatrix}
\frac{\partial u_1}{\partial S} & 0 & \frac{1}{S} \frac{\partial u_1}{\partial \theta} - \frac{w_1}{S} \\
0 & \frac{u_1}{S} + \frac{w_1}{S} \cot \theta & 0 \\
\frac{\partial w_1}{\partial S} & 0 & \frac{u_1}{S} + \frac{\partial w_1}{\partial \Phi}
\end{bmatrix}. \quad (8)$$

The solution of the second-order problem for an isotropic incompressible elastic material reduces to the solution of the equations

$$E^2 \psi_2 = \sin \theta \left\{ \frac{\partial}{\partial S} (SH_2) - \frac{\partial H_1}{\partial \theta} \right\}, \quad (9)$$

$$V^2 p_2 = -\left\{ \frac{\partial H_1}{\partial S} + \frac{2H_1}{S} + \frac{1}{S} \frac{\partial H_2}{\partial \theta} + H_2 \cot \theta \right\}, \quad (10)$$

where $\psi_2$ is the second-order displacement function, $p_2$ is the second-order hydrostatic pressure and

$$H_1 = -\left\{ \frac{3P_i'}{S} + \frac{1}{S} \frac{3P_i' \theta}{\theta} + \frac{1}{S} \left[ 2P_i' S_s - P_i' \phi - P_i' \theta + P_i' \theta s \cot \theta \right] \right\},$$

$$H_2 = -\left\{ \frac{3P_i'}{S} + \frac{1}{S} \frac{3P_i' \theta}{\theta} + \frac{1}{S} \left[ (P_i' \phi - P_i' \theta) \cot \theta + 2P_i' S_s + P_i' \theta s \right] \right\}. \quad (11)$$
The terms $P_{ss}', \ldots, \text{etc.}$ are the elements of a non-symmetric matrix $P_2'$ given by

$$P_2' = \begin{pmatrix} P_{ss}' & 0 & P_{s\theta}' \\ 0 & P_{\phi\phi}' & 0 \\ P_{\theta s}' & 0 & P_{\theta\theta}' \end{pmatrix},$$  \hspace{1cm} (12)

and

$$P_2' = P_1 C_1 + \frac{1}{2} u \left[ D C_1 - C_1^2 \right] + \left( C^T \right)^2 + \frac{w_1}{S} (QC_1 - C_1 Q) - 2C_2 C_1^2. \hspace{1cm} (13)$$

In (13) the operator $D$ is given by

$$D = u_1 \frac{\partial}{\partial S} + \frac{w_1}{S} \frac{\partial}{\partial \theta},$$  \hspace{1cm} (14)

$Q$ denotes the skew-symmetric matrix

$$Q = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$  \hspace{1cm} (15)

and $C_2$ is a material constant. The second-order displacement components $u_2$ and $w_2$ can be represented in the form

$$u_2 = u_2' + u_2'', \hspace{0.5cm} w_2 = w_2' + w_2'', \hspace{1cm} (16)$$

where

$$u_2' = -\frac{1}{S^2 \sin \theta} \frac{\partial w_2}{\partial \theta}, \hspace{0.5cm} w_2' = \frac{1}{S \sin \theta} \frac{\partial w_2}{\partial \theta},$$  \hspace{1cm} (17)

and

$$u_2'' = \frac{1}{2} Du_1 - \frac{1}{2} \frac{w_2}{S}, \hspace{0.5cm} w_2'' = \frac{1}{2} Dw_1 + \frac{1}{2} \frac{u_1 w_1}{S}. \hspace{1cm} (18)$$

The second-order constitutive equation is

$$T_2 = -P_2 I + wC_2' + P_2' + \frac{w_1}{S} \left\{ T_1 Q - QT_1 \right\} + C_1 T_1,$$  \hspace{1cm} (19)

where $T_2$ is the second-order component of the Cauchy stress tensor and

$$G_2' = G_2' + G_2^T,$$  \hspace{1cm} (20)

where

$$G_2' = \begin{pmatrix} \frac{\partial u_2'}{\partial S} & 0 & \frac{1}{S} \frac{\partial u_2'}{\partial \theta} - \frac{w_2}{S} \\ 0 & \frac{u_2'}{S} + \frac{w_2'}{S} \cot \theta & 0 \\ \frac{\partial w_2'}{\partial \theta} & 0 & \frac{u_2'}{S} + \frac{1}{S} \frac{\partial w_2'}{\partial \theta} \end{pmatrix}. \hspace{1cm} (21)$$

To the second-order in terms of the non-dimensional parameter $\varepsilon$
Spherical inclusion subjected to a resultant force

Consider an incompressible infinite elastic medium which is bounded internally by a rigid spherical inclusion of radius $a$. The inclusion is in welded contact with the elastic medium. The inclusion is subjected to a central force ($F$) which causes a rigid body translation $\xi$. The cylindrical polar coordinate system of the reference configuration is chosen such that the $Z$-axis coincides with the line of action of $F$ (Fig.1).

It may be verified that the first-order displacement function

$$\psi_1 = \frac{\xi}{4a} \left\{ \frac{a}{S} - 3a^2 S \right\} \sin^2 \theta,$$  \hspace{1cm} (23)

satisfies displacement boundary conditions

$$u_1 = \xi \cos \theta, \quad w_1 = -\xi \sin \theta, \quad \text{on} \quad S = a, \quad (24)$$

and gives stress components which tend to zero as $S \to \infty$. The small real dimensionless parameter is chosen to be equal to $(\xi/4a)$. The first-order displacement and stress components can be written as

$$u_1^* = \left\{ -\frac{2}{S^{3/3}} + \frac{6}{S^{2/3}} \right\} \cos \theta, \quad w_1^* = \left\{ -\frac{1}{S^{2/3}} - \frac{3}{S^{1/3}} \right\} \sin \theta,$$ \hspace{1cm} (25)
\[ T_{ss}^{(1)*} = \left\{ \frac{12}{S^{*}} - \frac{18}{S^{**}} \right\} \cos \theta, \quad T_{\phi \phi}^{(1)*} = T_{\theta \theta}^{(1)*} = -\frac{6}{S^{**}} \cos \theta, \]
\[ T_{s \theta}^{(1)*} = \frac{6}{S^{**}} \sin \theta, \]

(26)

where \( ()^* \) denote the dimensionless variables which can be related to the physical variables according to the following:

\[ u_1 = au_1^*, \quad w_1 = aw_1^*, \quad T_1 = \mu T_1^*. \]

(27)

If the expressions (25) and (26) for the first-order displacement and stress components are substituted in (9), the inhomogeneous differential equation for the second-order displacement function \( \psi_2^* = \psi_2 / a^3 \) reduces to

\[ E^* \psi_2^* = 216 \left\{ \frac{4C_2}{\mu} - 1 \right\} \left\{ \frac{2}{S^{*}} - \frac{1}{S^{**}} \right\} \sin^2 \theta \cos \theta, \]

(28)

where

\[ E^* = \frac{\partial^2}{\partial S^{*2}} + \frac{1}{S^{**}} \frac{\partial^2}{\partial \theta^2} - \cot \theta \frac{\partial}{\partial \theta}. \]

(29)

A particular integral of (28) is

\[ \psi_2^* = \chi \left\{ \frac{3}{S^{*}} + \frac{\alpha}{S^{**}} \right\} \sin^2 \theta \cos \theta, \]

(30)

where \( \chi = (C_2 - C_1)/(C_2 + C_1) \), and \( C_1 \) is an elastic constant such that \( 2(C_1 + C_2) = \mu \). It can be verified that the second-order displacement components derived from (30) and (16) do not satisfy displacement boundary conditions \( u_2^*(1, \theta) = w_2^*(1, \theta) = 0 \), on the inclusion-elastic medium interface. These boundary conditions can be explicitly satisfied by employing solutions of the homogeneous equation \( E^* \psi_2^* = 0 \). The appropriate homogeneous solution is

\[ \psi_2^* = \left\{ \alpha_1 + \frac{\alpha_2}{S^{**}} \right\} \sin^2 \theta \cos \theta, \]

(31)

where \( \alpha_1, \alpha_2 \) are constants. The complete second-order displacement function is therefore

\[ \psi_2^* = \left[ \frac{2C_2}{\mu} \left\{ \frac{6}{S^{*}} - \frac{18}{S^{**}} + \frac{18}{S^{*}} - 6 \right\} + \left\{ -\frac{3}{S^{*}} + \frac{15}{S^{**}} - \frac{9}{S^{*}} - 3 \right\} \right] \sin^2 \theta \cos \theta. \]

(32)

The complete second-order displacement and stress components determined from (16), (19) and (32) are
$u_2^* = \left[ \frac{36C_2}{\mu} \left\{ -\frac{1}{S^{1/5}} + \frac{3}{S^{3/5}} - \frac{3}{S^{1/3}} + \frac{1}{S^{2/3}} \right\} + 
 + \left\{ -\frac{9}{2} \frac{1}{S^{1/5}} + \frac{36}{S^{3/5}} - \frac{45}{S^{4/5}} + \frac{9}{2} \frac{1}{S^{3/5}} + \frac{9}{2} \frac{1}{S^{2/3}} \right\} \right] \cos^2 \theta + 
 + \frac{12C_2}{\mu} \left\{ \frac{1}{S^{1/5}} - \frac{3}{S^{3/5}} + \frac{3}{S^{1/3}} - \frac{1}{S^{2/3}} \right\} + 
 + \left\{ -\frac{3}{2} \frac{1}{S^{1/5}} - \frac{6}{S^{3/5}} + \frac{15}{S^{4/5}} - \frac{9}{2} \frac{1}{S^{3/5}} - \frac{3}{S^{2/3}} \right\}, \quad (33)$

$w^* = \left[ \frac{36C_2}{\mu} \left\{ -\frac{1}{S^{1/5}} + \frac{2}{S^{3/5}} - \frac{1}{S^{3/3}} \right\} - 
 - \frac{3}{2} \frac{1}{S^{1/5}} + \frac{18}{S^{3/5}} - \frac{30}{S^{4/5}} + \frac{27}{2} \frac{1}{S^{3/3}} \right] \sin \theta \cos \theta,$

and

$T^{(2)*}_{ss} = \left[ \frac{72C_2}{\mu} \left\{ -\frac{1}{S^{1/5}} + \frac{7}{S^{3/5}} - \frac{12}{S^{4/5}} + \frac{9}{S^{5/5}} - \frac{3}{S^{1/3}} \right\} + 
 + \frac{84}{S^{5/5}} - \frac{333}{S^{5/5}} + \frac{360}{S^{3/5}} - \frac{54}{S^{1/3}} \right] \cos^2 \theta + 
 + \frac{12C_2}{\mu} \left\{ -\frac{10}{S^{1/5}} + \frac{24}{S^{3/5}} - \frac{21}{S^{4/5}} + \frac{6}{S^{5/5}} \right\} + 
 + \frac{9}{S^{3/3}} + \frac{3}{S^{3/3}} + \frac{120}{S^{5/5}} + \frac{36}{S^{3/3}} + \frac{18}{S^{3/3}}, \quad (33)$

$T^{(2)*}_{\phi\phi} = \left[ \frac{72C_2}{\mu} \left\{ \frac{1}{S^{1/5}} - \frac{6}{S^{3/5}} + \frac{5}{S^{4/5}} - \frac{1}{S^{5/5}} \right\} + 
 + \left\{ -\frac{21}{S^{1/5}} + \frac{135}{S^{3/5}} - \frac{150}{S^{4/5}} + \frac{45}{S^{5/5}} \right\} \right] \cos^2 \theta + 
 + \frac{12C_2}{\mu} \left\{ \frac{6}{S^{1/5}} + \frac{2}{S^{3/5}} - \frac{6}{S^{4/5}} + \frac{3}{S^{5/5}} \right\} + 
 + \left\{ -\frac{12}{S^{1/5}} - \frac{15}{S^{3/5}} + \frac{30}{S^{4/5}} - \frac{9}{S^{5/5}} \right\}, \quad (33)$

$T^{(2)*}_{\theta\theta} = \left[ \frac{72C_2}{\mu} \left\{ \frac{2}{S^{1/5}} - \frac{7}{S^{3/5}} + \frac{7}{S^{4/5}} - \frac{2}{S^{5/5}} \right\} + 
 + \left\{ -\frac{45}{S^{1/5}} + \frac{117}{S^{3/5}} - \frac{210}{S^{4/5}} + \frac{63}{S^{5/5}} \right\} \right] \cos^2 \theta + 
 + \frac{12C_2}{\mu} \left\{ \frac{8}{S^{1/5}} - \frac{14}{S^{3/5}} + \frac{9}{S^{4/5}} \right\} + \frac{12}{S^{1/5}} + \frac{3}{S^{3/5}} + \frac{90}{S^{4/5}} - \frac{27}{S^{5/5}}, \quad (34)$

$T^{(2)*}_{s\theta} = \left[ \frac{72C_2}{\mu} \left\{ -\frac{1}{S^{1/5}} + \frac{5}{S^{3/5}} - \frac{6}{S^{4/5}} + \frac{5}{S^{5/5}} - \frac{1}{S^{1/3}} \right\} + 
 + \frac{48}{S^{1/5}} - \frac{126}{S^{3/5}} + \frac{240}{S^{4/5}} - \frac{144}{S^{5/5}} - \frac{18}{S^{1/3}} \right] \sin \theta \cos \theta, \quad (34)$
respectively.

**Force-displacement relationship for the spherical inclusion**

Consider a closed spherical surface $\partial \Sigma$ in the deformed body. The component of the resultant force in the $X_3$ direction, over $\partial \Sigma$ is given by

$$F_z = \int_{\partial \Sigma} \left\{ T_{ss} \cos \theta - T_{s\theta} \sin \theta \right\} s_0 \sin \theta \, d\theta,$$  \(35\)

where $s_0, \theta$ are the coordinates of a particle on $\partial \Sigma$. Also on $\partial \Sigma$

$$s_0 = S + \varepsilon u_1(s_0, \theta), \quad \theta = \theta + \varepsilon \frac{\partial w}{\partial S}(s_0, \theta),$$  \(36\)

to order $\varepsilon$; and to this order $S$ may be replaced by $s_0$. By considering the series expansions for $T_{ss}$ and $T_{s\theta}$, (35) can be reduced to a form

$$F_z = \varepsilon F_z^{(1)} + \varepsilon^2 F_z^{(2)},$$  \(37\)

where

$$F_z^{(1)} = \int_{\partial \Sigma} \left\{ T_{ss}^{(1)} \cos \theta - T_{s\theta}^{(1)} \sin \theta \right\} s_0 \sin \theta \, d\theta,$$  \(38\)

and

$$F_z^{(2)} = \int_{\partial \Sigma} \left\{ T_{ss}^{(2)} \cos \theta - T_{s\theta}^{(2)} \sin \theta - \frac{\partial w}{\partial S} \right\} \left\{ T_{ss}^{(1)} \sin \theta + T_{s\theta}^{(1)} \cos \theta \right\} +$$

$$- u_1 \left\{ \frac{\partial T_{ss}^{(1)}}{\partial S} \cos \theta - \frac{\partial T_{s\theta}^{(1)}}{\partial S} \sin \theta \right\} +$$

$$+ \left\{ T_{ss}^{(1)} \cos \theta - T_{s\theta}^{(1)} \sin \theta \right\} \left\{ \frac{1}{S} \frac{\partial w}{\partial \theta} + \frac{\partial w}{\partial S} \cot \theta \right\} \right\} s_0 \sin \theta \, d\theta.$$  \(39\)

Evaluating the integrals (38) and (39) we obtain

$$\varepsilon F_z^{(1)} = -6\pi u_1 \xi,$$  \(40\)

$$F_z^{(2)} = 0.$$  \(40\)

The result $F_z^{(2)} = 0$ confirms the fact that from the particular spatial symmetry of the problem the force $F$ must be an odd function of the inclusion displacement $\xi$. Therefore the result $F = -6\pi u_1 \xi$ is valid to order $\varepsilon^3$.

**References**