Some Easy-to-Implement Methods of Calculating American Futures Option Prices

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Options on various types of futures contracts (stock index, interest rate, currency, energy, metal and agricultural commodities) are currently traded on major exchanges across the world. By and large, these options are of the American type, i.e., they can be exercised any time prior to the option's maturity date. Black's (1976) European futures option formula ignores this early exercise possibility and hence underestimates the value of these options. Despite this obvious shortcoming, practitioners continue to use Black's European formula for many purposes, for example, to estimate American futures option values, to calculate margin requirements, and so forth [Wolf (1984)]. Because of its analytic and closed form nature, Black's formula is intuitively appealing and extremely user friendly. In these respects, the extant methods of American futures option valuation are, however, deficient although they yield more accurate option values.

Building on some recent works [Lieu (1990); Chaudhury and Wei (1994); Chen and Scott (1993)], this article presents a new approach to the valuation of American futures options. The American futures option value (early exercise premium) is shown to be a multiple (proportion) of Black's (1976) European futures option value. Aside from providing new

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economic insights about the value of the early exercise privilege and the marking-to-market feature, the new valuation approach paves the way for the four alternative methods of calculating American futures option prices presented in this article. Each of these methods approximates the multiple applied to Black's European futures option value in a different way.

While the proposed methods are not exact solutions to the American futures option valuation problem, they are fairly accurate and yet share the closed form, analytic, and simplistic nature of Black's (1976) European futures option valuation formula. As shown by the simulation results, what is gained relative to Black's European formula is a level of accuracy which is at least as good as that offered by the quadratic approximation method [Barone-Adesi and Whaley (1987), BW], which is computationally the most efficient of the existing American futures option valuation methods.

Option pricing models are used to estimate ISD or market participants' ex ante assessment of futures (and spot) price volatility implied by observed option prices. Because of the liquidity of futures option markets and the fact that yield on the spot asset is not required as an input for futures option pricing, futures options often provide a better venue for ISD estimation than spot options. ISD estimation using the existing methods of American futures option valuation requires a joint iterative search for ISD and the critical futures price(s) above/below which an American futures call/put option would be exercised. This can easily become an unmanageable task [Shastri and Tandon (1986c, p. 382)]. This is perhaps another reason why Black's European formula is preferred by many practitioners although an overestimation of ex ante volatility is expected theoretically.[1] Estimation of ISD using the valuation methods proposed in this article is similar to employing Black's European formula. What is avoided, however, is the upward bias in the ISD estimate imparted by Black's formula and the complication of a joint search for ISD and critical futures price(s) required by the current American futures option valuation methods.

The methods proposed in this article should appeal to a wide spectrum of users. People involved in the production and processing of agricultural commodities and metals, investors, local traders, exchange officials and brokers who have traditionally used Black's European

[1]While analytic approximations [Barone-Adesi and Whaley (1987); Geske and Johnson (1984)] have been used by some empirical researchers [e.g., Cakici, Chatterjee, and Wolf (1993); Shastri and Tandon (1986c); Whaley (1986)], others [e.g., Ball and Torous (1986); Wilson and Fung (1990); Wilson, Fung, and Ricks (1988)] continue to use Black's European formula.

formula as a valuation guide (to American futures options) for the purposes of ISD estimation, trading in futures and futures options, setting and calculating margin requirements can achieve greater accuracy with no more computational burden using these methods. On the other hand, practitioners and researchers who use the extant American futures option valuation methods because of their greater accuracy should find these methods useful as well. They are easier to use with only a marginal loss in accuracy. In fact they are more accurate in some cases.

The remainder of this article is organized as follows. In the next section, the existing methods of futures option valuation are reviewed. A new approach to American futures option valuation is developed in the second section. Based upon this new valuation approach, four alternative methods of calculating American futures option value are then presented followed by the simulation results.

CURRENT STATE OF FUTURES OPTION VALUATION

The following notation is used for expositional purposes:

t = time subscript

F = current ($t = 0$) futures price

X = striking price of futures option

T = expiration time of futures option

r = constant risk-free rate of interest

$C(c)$ = American (European) futures call option price

$P(p)$ = American (European) futures put option price.

A common assumption in the option-pricing literature is that futures prices have a lognormal distribution, i.e., follow the stochastic differential equation,

$$dF/F = a\,dt + s\,dz \tag{1}$$

where a is the expected instantaneous price change relative of futures contract, s is the instantaneous standard deviation, and z is a Weiner process.

Black's European Futures Option Valuation Formula

Using standard frictionless market assumptions, a constant riskless rate, r, and arbitrage-free conditions, Black (1976) demonstrates that the futures option price would follow the differential equation,

$$0.5s^2F^2V_{FF} - rV + V_t = 0 \tag{2}$$

where V refers to C, c, P, or p. Using suitable boundary conditions, Black derives analytic solutions for European futures option prices:[2]

$$c = \exp(-rT)[FN(d_1) - XN(d_2)] \tag{3}$$

$$p = \exp(-rT)[XN(-d_2) - FN(-d_1)] \tag{4}$$

where $d_1 = [\ln(F/X) + 0.5s^2T]/sT^{0.5}$, and $d_2 = d_1 - sT^{0.5}$ and $N(\cdot)$ is the cumulative standard normal distribution function.

Exchange-traded options are by and large of the American type. Ramaswamy and Sundaresan (1985), Brenner, Courtadon, and Subrahmanyam (1985), and Ball and Torous (1986) show that, for positive interest rates, an American futures option may be exercised prior to maturity even in the absence of payout from the spot asset underlying the futures contract. Because of this early exercise possibility, an American futures option would trade at a premium relative to its European counterpart. Since Black's European formulas in (3) and (4) do not incorporate this early exercise premium, using them would lead to an underestimation of American futures option values.[3] Also, as option values increase with volatility, the implied standard deviation from Black's formulas would overestimate the market participants' ex ante assessment of futures (and spot) price volatility.

Numerical Methods of American Futures Option Valuation

Numerical methods to calculate American futures option values can be classified into two broad categories: the lattice approach [Cox, Ross, and Rubinstein (1979); Cox and Rubinstein (1985; Parkinson (1977)] and the finite difference approach [Brennan and Schwartz (1977,

[2]Strictly speaking, Black's (1976) European formula applies to options on forward contracts. However, given the assumption of a constant risk-free rate, forward and futures prices would be equal; hence, options on forward and futures contracts would have the same value.

[3]The pricing bias may be small in some cases [Shastri and Tandon (1986a, 1986b)].

1978); Brenner, Courtadon, and Subrahmanyam (1985); Ramaswamy and Sundaresan (1985); Schwartz (1977)].[4] Both categories of numerical methods split the time to maturity into small intervals. Starting at maturity, they compute the option value over each small interval of time, applying the American option boundary condition at every step, and dynamically work backwards through time to arrive at the current option value. The two categories of numerical methods differ in how they calculate live American option value over the small time intervals.[5]

The strongest point of the numerical methods is that they lead to quite accurate option values. In fact, the numerical method prices are taken often to be the *true* option prices for simulation purposes. However, to ensure a high level of accuracy using the numerical methods, a main frame computer is usually needed [BW (1987, p. 304)]. The numerical methods are not easily amenable to comparative statistics and they lack the intuitive appeal of Black's European formula. They are also quite cumbersome for ISD estimation since the intermediate option values at every discretized time interval depend upon the futures price volatility.

Analytic Solution

An analytic solution to the American futures option valuation problem does exist in the literature [Kim (1994)].[6]

$$C = c + \int_0^T r \exp(-rt)[FN(d_{1t}) - XN(d_{2t})]\,dt \qquad (5)$$

$$P = p + \int_0^T r \exp(-rt)[XN(-d_{2t}) - FN(-d_{1t})]\,dt \qquad (6)$$

where $d_{1t} = [\ln(F/G(t)) + 0.5s^2t]/st^{0.5}$, $d_{2t} = d_{1t} - st^{0.5}$, $G(t) = F^*(t)$, $t = 0, \ldots, T$, is the optimal exercise boundary for American

[4]Review of the numerical methods can be found in Courtadon (1990), Geske and Shastri (1985), and Hull (1989).

[5]The lattice approach approximates the stochastic process governing the optioned asset's price by a discrete statistical distribution (e.g., binomial, trinomial). Option values at each step are then calculated by risk-neutral discounting of the expected payoff over the next time interval. The finite difference approach, on the other hand, approximates the continuous time partial derivatives in (2) and solves for option values at each step so as to satisfy (2) over the next small interval of time.

[6]Kim's (1994) exact analytic solution for American futures options is a special case of his earlier work (Kim, 1990) on the general problem of American option valuation. Similar solutions were derived separately by Jacka (1991) and Carr, Jarrow, and Myneni (1992) for American put options on stocks, and by Jamshidian (1992) for American call and put options in general.

futures call options, and $G(t) = F^{**}(t)$, $t = 0, \ldots, T$, is the optimal exercise boundary for American futures put options.

As shown by Kim (1990), the above solutions can be implemented by splitting the time to maturity into M small intervals, numerically solving M integral equations for as many critical futures prices in a recursive fashion (starting just before maturity), and performing numerical integration using those critical futures prices. To simplify Kim's (1990) procedure, Kim (1994) suggests an approximate functional form for the optimal exercise boundary which has two unknown parameters. Using simulation data, Kim (1994) estimates polynomial regression functions to predict the unknown parameters.

Kim's (1994) implementation of analytic solution offers a more computationally efficient analytic alternative to the numerical methods.[7] It is also accurate for longer maturity options where BW's quadratic approximation performs poorly. For shorter maturity options, however, it does not offer any practical advantage over BW's method in calculating American futures option prices.[8] Kim's procedure requires calculation of a large number of critical futures prices compared to just one in BW's method. Further, because of the requirement of numerical integration of analytic solution, Kim's solution is not particularly suitable for ISD estimation.

Analytic Approximation: Compound Option Method

Geske and Johnson's (1984) approach, adapted by Shastri and Tandon (1986b, 1986c) to the case of American futures options, expresses the American futures option as an infinite series of European compound options. The formula is implemented using Richardson's extrapolation technique to approximate the infinite series by only a few (three to four) terms (compound options).

Geske and Johnson's compound option approach is about 20 times more computationally efficient than numerical methods (BW (1987, p. 304)], primarily due to the calculation of only a few (three to six) critical futures prices. However, it requires the evaluation of cumulative bivariate, trivariate, and sometimes higher order multivariate normal

[7] Kim (1994, p. 13) reports that for penny accuracy his procedure is about one hundred times faster than the implicit finite difference method.

[8] For penny accuracy in calculating the option price, Kim's (1994) procedure requires 20% more computer time than BW's quadratic approximation. Kim's procedure is, however, more accurate than BW's method in calculating hedge ratios.

density functions. As concluded by BW, there is no practical advantage in using the compound option method (rather than BW's method) for shorter maturity options. While its performance is better than BW's method for longer maturity options, the performance of both Kim and Geske and Johnson's methods deteriorates for shorter maturity options. Kim's (1994) approximation is more accurate than Geske and Johnson's method and no iterative search for the critical futures prices is needed.

Analytic Approximation: Quadratic Approximation Method

The second term in eqs. (5) and (6) above represents the exact functional form for the early exercise premium of American futures call and put options. Building on MacMillan's (1986) work, BW derived a quadratic approximation to the early exercise premium of American futures options which leads to the following formulas for American futures option values:

$$\begin{aligned} \mathrm{CBW} &= c + A_2(F/F^*)^{q2}, \qquad \text{for } F < F^*, \\ \text{and} \quad C &= F - X, \qquad \text{for } F \geq F^* \end{aligned} \tag{7}$$

$$\begin{aligned} \mathrm{PBW} &= p + A_1(F/F^{**})^{q1}, \qquad \text{for } F > F^{**}, \\ \text{and} \quad P &= X - F, \qquad \text{for } F \leq F^{**} \end{aligned} \tag{8}$$

where

$$\begin{aligned} A_2 &= (F^*/q_2)\{1 - \exp(-rT)N[d_1(F^*)]\}, \\ d_1(F^*) &= [\ln(F^*/X) + 0.5s^2T]/sT^{0.5}, \\ q_2 &= [1 + (1 + 4k)^{0.5}]/2, \qquad k = 2r/\{s^2[1 - \exp(-rT)]\}, \\ A_1 &= -(F^{**}/q_1)\{1 - \exp(-rT)N[-d_1(F^{**})]\}, \\ d_1(F^{**}) &= [\ln(F^{**}/X) + 0.5s^2T]/sT^{0.5}, \\ q_1 &= [1 - (1 + 4k)^{0.5}]/2, \qquad \text{and} \end{aligned}$$

F^* (F^{**}) is the critical futures price, i.e., the price above (below) which the American call (put) option should be exercised.

The quadratic approximation is simpler and computationally more efficient than other American futures option pricing methods [numerical; Geske and Johnson (1984); Kim (1990, 1994)], yet it provides good accuracy for shorter maturity options. While quadratic approximation involves just the current critical futures price, an interactive search for this critical futures price level is still required.[9] For this reason, ISD estimation using quadratic approximation remains a more involved process than using Black's European formula. Further, the pricing accuracy of quadratic approximation weakens considerably for longer maturity futures options [BW (1987, p. 316); Kim (1994)]. Kim (1994, p. 8) indicates that the exchange-traded and OTC longer term options are gaining importance.

This article proposes a new approach to American futures option valuation that is as user friendly as Black's European formula and is as accurate as the quadratic approximation method.

A NEW APPROACH TO AMERICAN FUTURES OPTION VALUATION

Pure Option Pricing

In a recent article, Lieu (1990) used arbitrage-free conditions similar to Black and Scholes (1973) and Black (1976) to derive exact formula for the value of a futures-style (pure) European option on futures contract,[10]

$$\text{call: } c_p = FN(d_1) - XN(d_2) = \exp(rT)c \tag{9}$$

$$\text{put: } p_p = XN(-d_2) - FN(-d_1) = \exp(rT)p \tag{10}$$

Unlike conventional options, a pure option does not require the buyer to pay the option premium at the time of establishing a long position. Instead, as in the case of futures trading, both the buyer and the seller post margin deposits with a broker to cover a one-day move in the option premium [Asay (1982, p. 5)] and their positions are marked-to-market on a daily basis.[11]

[9]In a recent article, Allegretto, Barone-Adesi, and Elliott (1993) propose some modifications to the BW method for put options which improves accuracy of the critical price (of the underlying asset) and the option price. Unfortunately, the modifications require more rather than less computing capability (a work station) than the BW method. Instead of one, a system of three equations needs to be solved interactively. Additionally, a relaxation parameter needs to be estimated (on an ad hoc basis).

[10]The same solution for a pure call option was noted earlier by Asay (1982).

[11]Duffie and Stanton's (1992, p. 571) solution for continuously resettled futures option is similar to Lieu's (1990) solution.

The London International Financial Futures Exchange (LIFFE) and the Sydney Futures Exchange (SFE) are the only exchanges where options with futures-style margining are currently traded. LIFFE trades futures options on the British Pound, the three-month Eurodollar interest rate, the three-month Sterling interest rate, the long gilt, U.S. treasury bond, the German Government bond, and the Financial Time Stock Exchange 100 Index (FTSE). At SFE, futures options are available on All Ordinaries Share Price Index (AOI), 90-day bank bills, and three-year and ten-year treasury bonds [Brace and Hodgson (1991)]. All of these options are of the American type.

Lieu (1990) argues that the pure European option price never falls below its intrinsic value. Since a pure American option would be at least as valuable as a pure European option, the pure American option price never falls below the intrinsic value either. As such a pure American option would never be exercised early and its value should be the same as that of a pure European option. Hence, $C_p = c_p$ and $P_p = p_p$. Chen and Scott (1993) show that these results hold under a stochastic interest rate regime as well.

American Futures Option Pricing

Using rational pricing arguments [Merton (1973)], Chaudhury and Wei (1994) prove that the pure option price is in fact an upper bound for the conventional or stock-style American futures option price.[12,13] Since the European futures option price is a natural lower bound for the American futures option price, it can be argued that the American futures option price lies in between c and c_p for a call option, and in between p and p_p for a put option. That is,

$$C\Gamma_C = \exp(rT)c \tag{11}$$

$$P\Gamma_P = \exp(rT)p \tag{12}$$

[12]The pure option price provides a tighter upper bound than the futures price (call) or the striking price (put) for American futures options. Unfortunately, for spot or stock options, the European put option price with futures-style margining does not provide an upper bound for the conventional (stock-style margining) American put option price. In this case, the European put option price (and therefore the American put option price if the option is kept alive) may fall below the intrinsic value even with futures-style margining.

[13]Henceforth, a conventional or stock-style futures option is referred to as simply *futures option*.

where $\exp(rT) > \Gamma_C > 1$, and $\exp(rT) > \Gamma_P > 1$. Omitting the subscript, $\exp(rT) > \Gamma$ restricts the American futures option price to be above the European futures option price; $\Gamma > 1$ imposes the pure price as an upper bound for the American futures option price. Chen and Scott's (1993, p. 21) example of Eurodollar American futures call options traded on the LIFFE (pure or futures-style) and the Chicago Mercantile Exchange (conventional or stock-style) shows that the American futures option price is indeed bounded by the European futures option price and the pure option price.

The multiplicative factor Γ can be interpreted as an interest savings factor. If an investor takes a long position in a pure European option as opposed to paying up front for a European futures option, the European futures option premium dollars would be freed up till maturity. As a result, the investor would save interest ($c[\exp(rT) - 1]$ or $p[\exp(rT) - 1]$) on the European futures option premium over the full maturity of the option with certainty. Hence, Γ is equal to $\exp(rT)$ for a pure European option.

Now consider a long position in a pure American option instead of paying up front for an American futures option. If the American futures option remains alive till maturity, the long position in the pure American option would save interest on the American futures option premium over full maturity; thus, the interest savings from taking a pure American option position would amount to $C[\exp(rT) - 1]$ or $P[\exp(rT) - 1]$. However, this maximum amount of interest savings is not guaranteed. If, prior to maturity, it turns out that the American futures option should be exercised early, the buyer of the pure American option has, in effect, saved interest on the initial American futures option premium (C, P) only until the time of premature exercise, not over full maturity. Due to this early exercise possibility, the expected life of an American futures option is less than its maturity. Accordingly, the associated Γ becomes an expected, rather than a certain, interest savings factor assuming values less than the maximum interest savings factor, $\exp(rT)$. Thus, although the pure American option buyer saves interest on a greater amount of capital $(C > c, P > p)$, the expected interest savings factor $\Gamma(< \exp(rT))$ is such that the value of a pure American option is the same as that of a pure European option.

The valuation link [eqs. (11) and (12)] between a pure (American and European) option and an American futures option, and the interpretation of Γ as an expected interest savings factor lay the grounds for a new approach to American futures option valuation. A rearrangement of (11) and (12) leads to the American futures option value expressed as a multiple of the European futures option value:[14]

$$C = [(1/\Gamma_C)\ \exp(rT)]c \tag{13}$$

$$P = [(1/\Gamma_P)\ \exp(rT)]p \tag{14}$$

Equations (13) and (14) provide American futures option prices when the current futures price is in the continuation range, i.e., when the options are alive. In the stopping region, of course, respective intrinsic values are the appropriate prices.

In the exact analytic solutions (5) and (6), the American futures option price is expressed as the sum of the European futures option price and the early exercise premium (EEP). Equations (13) and (14) show that the early exercise premium is, in fact, proportional to the European futures option price. These proportions are:

$$EEP_C/c = [\exp(rT) - \Gamma_C]/\Gamma_C \tag{15}$$

$$EEP_P/p = [\exp(rT) - \Gamma_P]/\Gamma_P \tag{16}$$

Accordingly, the early exercise premium expressions are:

$$EEP_C = c[\exp(rT) - \Gamma_C]/\Gamma_C \tag{17}$$

$$EEP_P = p[\exp(rT) - \Gamma_P]/\Gamma_P \tag{18}$$

As early exercise of an American futures option becomes a remote possibility, the buyer of a corresponding pure American option can expect to save interest on the conventional premium over almost the full maturity. Thus, Γ approaches its upper bound $\exp(rT)$ and the early exercise premium becomes negligible. In this scenario, the American futures option behaves like its European counterpart.

If, on the other hand, early exercise of an American futures option becomes imminent, the buyer of a pure American option cannot expect to save interest for very long. Hence Γ approaches its lower bound and

[14]Alternatively, the American futures option value may be viewed as a fraction $(1/\Gamma)$ of the pure option value. In other words, the American futures option value may be calculated by a downward adjustment of the pure option value.

the early exercise premium is significant both in dollar terms and as a proportion of European price.[15]

The expected interest savings factor Γ is inversely (directly) related to the probability of early exercise (expected life of the American futures option) and is a function of the current and future critical futures prices. The methods that are proposed in the next section attempt to estimate Γ without calculating the critical futures price levels. Once Γ is estimated, calculation of American futures option price using (13) and (14) becomes as easy as calculating European futures option price using Black's formula.

The pure option pricing results and the American futures option valuation approach presented above together offer some interesting economic insights.

First, while it is implicit in Chaudhury and Wei's (1994) results, the early exercise premium expressions (17) and (18) explicitly show that the interest savings generated by futures-style margining of a European futures option position, $c[\exp(rT) - 1]$ or $p[\exp(rT) - 1]$, is an upper bound for the early exercise premium of an American futures option.

Second, the same upper bound can be viewed also as the value of the marking-to-market of a European futures option position. Thus, futures-style margining or marking-to-market a European futures option position is more valuable than the addition of the early exercise feature.

Marking-to-market a European futures option position frees up capital equivalent to the current European futures option premium for the full maturity with certainty. The early exercise feature, on the other hand, frees up capital equivalent to intrinsic value if and when the American futures option is exercised. Although a greater amount of capital would be released upon exercise, early exercise is by no means guaranteed if the American futures option is currently alive.

Third, when an American futures option goes deep-in-the-money, its value changes almost dollar-for-dollar as the futures price changes. Since no capital is required for the futures position, the optionholder can enjoy the same futures price change by exercising the option (resulting in a futures position) and thus having the option premium freed up on which risk-free interest can be earned. In this situation, marking-to-market the American futures option would be of little additional value.

[15] Strictly speaking, the lower bound for Γ is greater than 1.0 since the pure option value remains above the intrinsic value even when the conventional American futures option is about to be exercised.

In essence, the early exercise feature of an American futures option serves the same capital-preserving role as implied by the marking-to-market mechanism if and when early exercise becomes imminent. In other situations, of course, the marking-to-market mechanism does add value since it frees up capital immediately while the early exercise feature bears only an uncertain (but probable) prospect of freeing up capital in the future.

Thus, intuitively speaking, marking-to-market a European futures option is equivalent to the addition of an early exercise feature (leading to an American futures option and bearing the prospect of capital release in early exercise situations) and marking-to-market an American futures option (freeing up capital when early exercise is not optimal). This can be seen by a little manipulation of (11) and (12):

$$c[\exp(rT) - 1] = EEP_C + C[\Gamma_C - 1] \tag{19}$$

$$p[\exp(rT) - 1] = EEP_P + P[\Gamma_P - 1] \tag{20}$$

One implication of (19) and (20) is that marking-to-market an American futures option is not the same as marking-to-market the European futures option and the early exercise option separately. In fact, marking-to-market an American futures option is less valuable than marking-to-market a European futures option. The reason is that interest savings over full maturity on a European futures option is guaranteed when it is marked-to-market by itself, but the interest savings on the European component are uncertain when an American futures option is marked-to-market. (When the early exercise privilege is used, the European futures option is surrendered too).

NEW METHODS

All four methods of calculating American futures option price that are about to be presented use eq. (13) for a call option and eq. (14) for a put option. Their difference lies in estimating the expected interest savings factor (EISF), Γ and, thus, the multiple that applies to Black's (1976) European futures option value.

With constant risk-free rate, $\Gamma = E[\exp(ry)]$, where E is the expectation operator and y is the actual remaining life of the option which can be observed ex post only. There are at least two complexities to deal with in evaluating Γ. First, the expectation E is of a nonlinear function of the random variable y. Second, the probability distribution of y is a function of the path of critical futures price and involves conditional probabilities. Consider, for example, an American call option with 1.0 year to maturity. That its actual life would be 1.0 year is not a sure thing and is conditional on the fact that it will not be exercised early. Thus, $\exp(ry)$ is equal to $\exp(r)$ with a conditional probability of no early exercise at $t = 0.0$, $t = 0.1$, $t = 0.2$, and so forth.

It appears that to be exact about Γ using y as the random variable, one would have to fall back on an approach similar to the compound option method. That is not a viable alternative given the objective of simplicity (as well as accuracy).

The task of evaluating Γ becomes a bit simpler if the life of the option is kept fixed at T and if the rate of interest savings (R) is treated as a random variable at various instants over the assumed full life of the option. However, it is not intended that option valuation occur in a stochastic interest rate environment. The risk-free rate is assumed to be constant. The variable is whether the risk-free rate can be earned over the next instant or not at various points in time. This depends on whether the American futures option remains alive or not.[16]

If the futures price is in the continuation range, the American futures option would remain alive and the buyer of a pure American option would save interest on the American futures option premium at the risk-free rate over the next instant. If, on the other hand, the futures price is in the stopping region, the buyer would save no interest since the American futures option would be rationally exercised.

Hence, in the continuation range of the futures price, the instantaneous rate of interest savings is equal to the risk-free rate ($R = r$). In the stopping region, the instantaneous rate of interest savings is zero ($R = 0$). Thus,

$$\Gamma = E\left[\exp\left(\int_0^T R(t)\,dt\right)\right] = E\left[\exp\left(r\int_0^T u(t)\,dt\right)\right] \tag{21}$$

where $u(t) = 1$ when the risk-free rate can be earned, and $u(t) = 0$ otherwise.

To handle nonlinearity of the function inside the expectation operator, a second-order Taylor series expansion of eq. (11) around

[16]This formulation is similar in spirit to the economic interpretation of the early exercise premium in the analytic solution. There, the early exercise premium is interpreted as the discounted value of the interests (on the intrinsic value) that the optionholder can expect to earn whenever the futures price enters the stopping region, i.e., crosses the optimal exercise boundary.

$x = E(x)$ leads to,

$$\Gamma \approx \exp\{E(x)\}[1 + 0.5 \operatorname{Var}(x)] \tag{22}$$

where

$$x = r \int_0^T u(t)\, dt$$

All four of the proposed methods (MA, MB, MC, MR) involve the first term in the above expansion. The second term involving the variance appears in only one of them (MC).

New Method: MA

The starting point for this as well as the other three new methods is to view $u(t)$'s, $t = 0, \ldots, T$, as a series of independent binomial variables with probability $n(t)$ for $u(t) = 1$:

$$\text{call: } n(t) = N\{-d^*(t)\} \tag{23}$$

$$\text{put: } n(t) = N\{d^{**}(t)\} \tag{24}$$

where $d^*(t) = [\ln\{F/F^*(t)\} - 0.5s^2t]/st^{0.5}$, $d^{**}(t) = [\ln\{F/F^{**}(t)\} - 0.5s^2t]/st^{0.5}$, $F^*(t)$ and $F^{**}(t)$ are the respective critical stock prices at time t. Given the current $(t = 0)$ futures price, F, $n(t)$ represents the probability that $F(t)$ would be in the continuation region i.e., the probability of no early exercise at time t. Since $u(t)$ is a binomial variable, $n(t) = E[u(t)]$. The buyer of a pure option can expect to save interest at the instantaneous rate, $r\ n(t)$, at a given point in time, t, during the life of the option. Hence,

$$E(x) = r \int_0^T n(t)\, dt$$

The main problem here is that $n(t)$'s cannot be exactly evaluated unless one knows the critical futures prices over the life of the American futures option. Additionally, integration of the cumulative normal probability will have to be done numerically. Therefore, some reasonable approximating structure needs to be imposed upon the $n(t)$'s to satisfy the objective of simplicity.

For the simplest method MA, it is assumed that $n(t)$ moves linearly through time:

$$n(t) = n(0) + [n(T) - n(0)]t/T \tag{25}$$

Since the critical futures price at time T is X, one knows that $n(T) = N(-d_2)$ for the call option and $n(T) = N(d_2)$ for the put option. Assuming that the option is currently alive, $n(0) = 1$, and using these end values, the following estimates of $E(x)$ are derived:

$$\text{call: } [1 - 0.5N(d_2)]rT$$

$$\text{put: } [0.5 + 0.5N(d_2)]rT$$

These, in turn, lead to the MA estimate for American futures call (CMA) and put (PMA) option values respectively:[17]

$$\text{CMA} = \text{Max}[F - X, c \exp\{0.5N(d_2)rT\}] \tag{26}$$

$$\text{PMA} = \text{Max}[X - F, p \exp\{0.5N(-d_2)rT\}] \tag{27}$$

New Method: MB

For the new methods, MB and MC, assume that $n(t)$ changes at the rate m in an exponential fashion:

$$n(t) = n(T) \exp\{-(T - t)m\} \tag{28}$$

Using the end values, $n(0) = 1$ and $n(T) = N(-d_2)$, for American futures call option and $n(T) = N(d_2)$ for American futures put option, one derives:

$$\text{call: } m = \ln[N(-d_2)]$$

$$\text{put: } m = \ln[N(d_2)]$$

Using these rates, the time path of the probability of no early exercise is described as:

$$\text{call: } n(t) = a_c^t, \qquad \text{where } a_c = [N(-d_2)]^{1/T} \tag{29}$$

$$\text{put: } n(t) = a_p^t, \qquad \text{where } a_p = [N(d_2)]^{1/T} \tag{30}$$

[17]The European option value, the pure option value, the option values generated by the BW approach, and the new methods are referred to as alternative price *estimates*. The term *estimate* here is not to be interpreted in a sample estimate sense. The *estimates* in this article are all theoretical futures option prices, i.e., all parameter values are assumed to be known. However, none of the *estimates* is an exact solution to the American futures option pricing problem. Accordingly, if one uses any of them to generate American futures option price, the resulting price figure may deviate from the true American futures option price (calculated using the binomial approach). It is because of such possible deviation or approximation error, that the term *estimate* is used.

Accordingly, the following estimates of $E(x)$ are derived:[18]

$$\text{call: } E(x_c) = (a_c^T - 1)/\ln(a_c) = -rTN(d_2)/\ln[N(-d_2)] \quad (31)$$

$$\text{put: } E(x_p) = (a_p^T - 1)/\ln(a_p) = -rTN(-d_2)/\ln[N(d_2)] \quad (32)$$

The MB estimates are then formed as:[19]

$$\text{CMB} = \text{Max}[F - X, c \exp(rT\{1 + N(d_2)/\ln N(-d_2)\})] \quad (33)$$

$$\text{PMB} = \text{Max}[X - F, p \exp(rT\{1 + N(-d_2)/\ln N(d_2)\})] \quad (34)$$

New Method: MC

The MC method modifies the MB method by adding the variance-related term of the Taylor expansion in evaluating the expected interest savings factor, Γ. This requires estimate of var(x). In developing var(x), $u(t)$'s are assumed to be independent binomial variables; in other words, whether the futures price at time $t + k$ is in the stopping or continuation region is independent of whether the futures price at time t is in the stopping or continuation region. The var(x) estimates are then derived as the sum of (integral over time) the variances of independent binomial variables, $r\ u(t)$'s:

$$\text{var}(x) = r^2 \int_0^T n(t)[1 - n(t)]\,dt$$

More specifically:

$$\text{call: var}(x_c) = -r^2T[N(d_2)]^2/\ln[N(-d_2)]^2 \quad (35)$$

$$\text{put: var}(x_p) = -r^2T[N(-d_2)]^2/\ln[N(d_2)]^2 \quad (36)$$

Substituting for $E(x)$ from (31) and (32) and for var(x) from (35) and (36), in the Taylor series expansion (22) for the expected interest savings factor, the MC American futures option price estimates are derived:

$$\text{call: CMC} = \text{Max}[F - X, c \exp(rT\{1 + N(d_2)/\ln N(-d_2)\})/ \{1 - r^2T[N(d_2)]^2/2\ln[N(-d_2)]^2\}] \quad (37)$$

[18]These $E(x)$ values are non-negative since the natural log operation on the probabilities leads to negative denominator values.

[19]The expressions inside the exponential operator are less than rT since the natural log operation on the probabilities leads to negative values.

$$\text{put: PMC} = \text{Max}[X - F, p \exp(rT\{1 + N(-d_2)/\ln N(d_2)\})/\{1 - r^2T[N(-d_2)]^2/2 \ln[N(d_2)]^2\}] \quad (38)$$

New Method: MR

The above methods (MA, MB, MC) estimate the value of an American futures option alive under the assumption that the option is currently alive (current futures price is in the continuation region), i.e., $n(0) = 1$. Accordingly, relatively high values are implicitly assigned to the $n(t)$'s in the near future. This may not be reasonable for in-the-money options where the current futures price is not so distant from the critical level. Since the MA, MB, and MC price estimates do not use the current critical futures price as an input, an under-estimation of the option value may result. While such options are the least actively traded in practice, it is still desirable to have a simple price estimate which incorporates more accurately the varying probability of no early exercise in the near future. The new method, MR, is a step in this direction.

While it is possible to modify any of the three previous methods, the simplest one, the MA method, is illustrated here. Instead of assuming $n(0) = 1$, $n(0)$ is estimated through regression.

Before elaborating on the MR method, a description of the simulation environment used in this article is outlined. The following alternative values for the relevant parameters are chosen:

r: low (0.04, 0.06), moderate (0.08, 0.10), high (0.12, 0.14)

s: low (0.10, 0.13, 0.15), medium (0.20, 0.23, 0.25), high (0.30, 0.35, 0.40)

T: short (0.25, 0.35, 0.50), medium (0.75, 0.85, 1.00), long (1.50, 2.00, 3.00)

X: 100

F: call: deep-out-of-the-money (80, 85), out-of-the-money (90), at-the-money (95, 100, 105), in-the-money (110), deep-in-the-money (115, 120)

put: deep-out-of-the-money (120, 115), out-of-the-money (110), at-the-money (105, 100, 95), in-the-money (90), deep-in-the-money (85,80)

Combinations of the above alternative parameter values characterize a total of 3159 call options and 3159 put options. The true American futures option value (C, P) in each case is calculated using a 750-step binomial procedure. The BW or quadratic price estimates (CBW, PBW) are calculated using eqs. (7) and (8).[20] The European futures option values (c, p) are calculated using eqs. (3) and (4), while the pure option prices (c_p, p_p) are calculated from eqs. (9) and (10). Equations (26) and (27) provide the MA estimates (CMA, PMA); eqs. (33) and (34) yield the MB estimates (CMB, PMB); and the MC estimates (CMC, PMC) follow from eqs. (37) and (38).

To derive the MR estimates of American futures option prices, the implied expected interest savings factor, Γ, is first calculated for each option from a rearrangement of eqs. (11) and (12):

$$\Gamma_C = \exp(rT)c/C$$

$$\Gamma_P = \exp(rT)p/P$$

Assuming a linear structure of $n(t)$,

$$\Gamma_C = \exp[0.5rT\{n_C(0) + N(-d_2)\}] \tag{39}$$

$$\Gamma_P = \exp[0.5rT\{n_P(0) + N(d_2)\}] \tag{40}$$

A rearrangement of the above expressions leads to an implied $n(0)$ for each option:

$$n_C(0) = [2\ \ln(\Gamma_C)/rT] - N(-d_2) \tag{41}$$

$$n_P(0) = [2\ \ln(\Gamma_P)/rT] - N(d_2) \tag{42}$$

The implied expected interest savings factor can only be known exactly using methods such as the finite difference or binomial. Instead, the new method, MR, predicts the $n(0)$ value for an option using the parameters $r,\ s,\ t,\ X$, and F. That $n(0)$ is related to these parameters is evident. For call options, this relationship is estimated by OLS regression of $n_c(0)$ on four variables $(r, s, t, \ln(F/X))$ using all 3121 observations with non-zero American futures option values when rounded to two decimal places:

$$n_c(o) = 1.0038\ - 1.5122\ r + 0.3905\ s - 0.0813\ T - 0.3897\ \ln(F/X)$$

$$t - \text{ratio}\ (69.05)\ (-14.12)\ (10.04)(-18.10)\ (-13.30)$$

$$R^2 = 20.6\%\ F(4, 3116) = 202.33 \tag{43}$$

[20]A fortran program (secant method) is used to calculate the critical futures price iteratively. The tolerance level for convergence is set at 0.005.

A similar regression equation is estimated using the 3141 put options in this study that have non-zero American futures option values when rounded to two decimal places:

$$n_P(0) = 0.8763 \; - 1.9290\, r + 1.0438\, s - 0.0207\, T + 0.6131 \ln(F/X)$$
$$t - \text{ratio } (37.06)\; (-11.03)\; (16.47)\, (-2.82)\; (-12.89)$$
$$R^2 = 15.5\% \; F(4, 3136) = 143.26 \qquad (44)$$

Both regressions and all the coefficient estimates are significant at conventional significance levels. For call and put options alike, early exercise is more likely with high risk-free rates, long maturities, and low volatility futures contracts. As expected, the likelihood of early exercise is higher with deeper in-the-money (less out-of-the-money) options.

Substituting the predicted $n(0)$ values, $n_{Ce}(0)$ and $n_{Pe}(0)$,[21] from the regression eqs. (43) and (44) in eqs. (39) and (40) leads to prediction of the corresponding expected interest savings factors,

$$\Gamma_{Ce} = \exp[0.5rT\{n_{Ce}(0) + N(-d_2)\}] \qquad (45)$$
$$\Gamma_{Pe} = \exp[0.5rT\{n_{Pe}(0) + N(d_2)\}] \qquad (46)$$

These predicted Γ's are, in turn, inserted into eqs. (13) and (14) to arrive at the MR estimate of American futures option price:

$$CMR = \text{Max}[F - X, c \exp(rT[1 - 0.5\{n_{Ce}(0) + N(-d_2)\}])] \qquad (47)$$
$$PMR = \text{Max}[X - F, p \exp(rT[1 - 0.5\{n_{Pe}(0) + N(d_2)\}])] \qquad (48)$$

To derive the MR estimate of the value of an American futures call option, one would simply calculate n_{Ce} using eq. (43) and the relevant parameter values (r, s, T, F, X) and then plug the n_{Ce} value into the formula in (47). Similar steps using (44) and (48) would lead to the MR estimate of the value of an American futures put option. Of course,

[21] Since OLS regression does not restrict the predicted values to be between 0.0 and 1.0, 0.0 (1.0) is used as the predicted value of $n(0)$ whenever the regression prediction is *below* (*above*) 1.0.

one may attempt to improve the price estimates by re-estimating (43) and (44) using alternative functional forms, estimation procedures, and options.[22] Since the set of options used in this study cover practically all realistic values of parameters, eqs. (43) and (44) are quite efficient, as will be shown later by the results.

It is worth mentioning that a re-estimation of the regression eqs. (43) and (44) would be necessary only if either the structure of the stochastic process governing futures price movement is different from the one described in eq. (1) or the new parameter values are beyond the ranges considered in this study. Given the wide ranges of parameter values used in the simulations, a re-estimation of the regression eqs. (43) and (44) due to the latter reason is not likely for most futures options that are currently traded. A similar comment applies to the regression equations suggested by Kim (1994) for the estimation of approximate exercise boundary functions.

SIMULATION RESULTS

A total of 3121 and 3141 parameter (r, F, T, s) combinations for call and put options, respectively, are considered. A given parameter combination along with a common striking price of 100 represents a given American futures option. For each parameter combination or option, eight different price figures are computed—the true American price (using the Binomial method) and seven estimates of this price: the European price (c, p), the pure option price (c_p, p_p), the quadratic approximation or BW price (CBW, PBW), the MA estimate (CMA, PMA), the MB estimate (CMB, PMB), the MC estimate (CMC, PMC), and the MR estimate (CMR, PMR).

The BW Cases

Table I contains the eight American futures call option price figures for each of the 25 parameter combinations reported in Tables III and V of BW.[23] The corresponding figures for put options are reported in Table II.

[22]The relationship is expected to be nonlinear. Hence, two alternative equations are estimated, each with six instead of four regressors—one with square and cube of $\ln(F/X)$ as additional regressors, and the other with square and cube of T as additional regressors. Neither of these materially improve the fit and the resultant price estimates over the linear regression reported here.

[23]The binomial prices are almost exactly the same as the finite difference prices reported in BW. The same is true of the quadratic approximation prices.

TABLE I
Theoretical American Futures Call Option Values Using Binomial, Quadratic Approximation, and New Methods, MA, MB, MC, and MR [Excercise Price (X) = 100]

Option Parameters[a]				*Option Values*							
				European[b]	*Pure*[c]	*American*					
								New Methods			
						Binomial[d]	*Quadratic*[e]	*MA*[f]	*MB*[g]	*MC*[h]	*MR*[i]
r	s	T	F	c	c_p	C	CBW	CMA	CMB	CMC	CMR
0.08	0.20	0.25	80.00	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.04
			90.00	0.70	0.71	0.70	0.70	0.70	0.70	0.70	0.70
			100.00	3.91	3.99	3.92	3.93	3.93	3.93	3.93	3.93
			110.00	10.74	10.96	10.82	10.81	10.83	10.82	10.85	10.84
			120.00	19.75	20.15	20.03	20.02	20.00	20.00	20.03	20.00
0.12	0.20	0.25	80.00	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.04
			90.00	0.69	0.71	0.69	0.70	0.69	0.69	0.69	0.69
			100.00	3.87	3.99	3.89	3.90	3.90	3.90	3.90	3.90
			110.00	10.63	10.95	10.76	10.75	10.76	10.76	10.79	10.79
			120.00	19.55	20.15	20.01	20.00	20.00	20.00	20.00	20.00
0.08	0.40	0.25	80.00	1.16	1.18	1.16	1.17	1.16	1.16	1.16	1.16
			90.00	3.52	3.59	3.53	3.53	3.53	3.53	3.53	3.53
			100.00	7.81	7.97	7.83	7.85	7.85	7.84	7.85	7.85
			110.00	14.01	14.29	14.09	14.08	14.10	14.10	14.11	14.10
			120.00	21.71	22.15	21.87	21.86	21.88	21.88	21.92	21.89

0.08	0.20	0.50	80.00	0.30	0.31	0.30	0.30	0.30	0.30	0.30	0.30
			90.00	1.70	1.77	1.71	1.72	1.71	1.71	1.71	1.71
			100.00	5.42	5.64	5.46	5.48	5.47	5.47	5.47	5.48
			110.00	11.73	12.21	11.90	11.90	11.90	11.90	11.94	11.93
			120.00	19.91	20.72	20.36	20.33	20.26	20.26	20.38	20.33
0.08	0.20	3.00	80.00	3.79	4.82	3.98	4.20	3.89	3.88	3.89	3.98
			90.00	6.81	8.66	7.25	7.54	7.07	7.06	7.08	7.28
			100.00	10.82	13.75	11.70	12.03	11.39	11.37	11.43	11.79
			110.00	15.71	19.97	17.32	17.64	16.76	16.73	16.87	17.42
			120.00	21.35	27.14	24.02	24.30	23.05	23.01	23.30	24.05

[a]The notation in this column is as follows: r = risk-free rate of interest; s = standard deviation of the commodity futures price change relative; T = time to maturity of the futures option; and F = current futures price.

[b]Values are calculated using eq. (3).

[c]Values are calculated using eq. (9).

[d]Values are computed using the binomial method with 750 time steps.

[e]Values are computed using the quadratic approximation eq. (7).

[f]Values are calculated using eq. (26).

[g]Values are calculated using eq. (33).

[h]Values are calculated using eq. (37).

[i]Values are calculated using eq. (43) and (47).

TABLE II

Theoretical American Futures Put Option Values Using Binomial, Quadratic Approximation, and New Methods, MA, MB, MC, and MR [Exercise Price (X) = 100]

Option Parameters[a]				*Option Values*							
				European[b]	*Pure*[c]	*American*					
								New Methods			
						Binomial[d]	*Quadratic*[e]	*MA*[f]	*MB*[g]	*MC*[h]	*MR*[i]
r	s	T	F	p	p_p	P	PBW	PMA	PMB	PMC	PMR
0.08	0.20	0.25	80.00	19.64	20.04	20.00	20.00	20.00	20.00	20.00	20.00
			90.00	10.50	10.71	10.59	10.58	10.59	10.59	10.62	10.61
			100.00	3.91	3.99	3.92	3.93	3.93	3.93	3.93	3.93
			110.00	0.94	0.96	0.94	0.94	0.94	0.94	0.94	0.94
			120.00	0.14	0.14	0.14	0.15	0.14	0.14	0.14	0.14
0.12	0.20	0.25	80.00	19.45	20.04	20.00	20.00	20.00	20.00	20.00	20.00
			90.00	10.40	10.72	10.53	10.53	10.53	10.53	10.57	10.57
			100.00	3.87	3.99	3.89	3.90	3.90	3.90	3.90	3.91
			110.00	0.93	0.96	0.93	0.93	0.93	0.93	0.93	0.93
			120.00	0.14	0.14	0.14	0.15	0.14	0.14	0.14	0.14
0.08	0.40	0.25	80.00	20.77	21.19	20.94	20.93	20.95	20.95	21.01	20.96
			90.00	13.32	13.59	13.39	13.39	13.42	13.42	13.44	13.42
			100.00	7.81	7.97	7.83	7.85	7.85	7.85	7.86	7.85
			110.00	4.21	4.30	4.22	4.23	4.22	4.22	4.22	4.22
			129,99	2.11	2.15	2.11	2.12	2.11	2.11	2.11	2.11

0.08	0.20	0.50	80.00	19.51	20.31	20.06	20.04	20.00	20.00	20.05	20.00
			90.00	11.31	11.77	11.48	11.48	11.49	11.49	11.53	11.52
			100.00	5.42	5.64	5.46	5.48	5.48	5.47	5.48	5.49
			110.00	2.13	2.22	2.14	2.15	2.14	2.14	2.14	2.14
			120.00	0.69	0.72	0.69	0.70	0.69	0.69	0.69	0.69
0.08	0.20	3.00	80.00	19.53	24.83	22.20	22.40	21.48	21.43	21.96	22.18
			90.00	14.67	18.65	16.21	16.50	15.93	15.89	16.14	16.30
			100.00	10.82	13.75	11.70	12.03	11.58	11.56	11.67	11.77
			110.00	7.85	9.98	8.37	8.69	8.29	8.28	8.32	8.37
			120.00	5.62	7.14	5.93	6.22	5.87	5.86	5.88	5.88

[a]The notation in this column is as follows: r = risk-free rate of interest; s = standard deviation of the commodity futures price change relative; T = time to maturity of the futures option; and F = current futures price.
[b]Values are calculated using eq. (4).
[c]Values are calculated using eq. (10).
[d]Values are computed using the binomial method with 750 time steps.
[e]Values are computed using the quadratic approximation eq. (8).
[f]Values are calculated using eq. (27).
[g]Values are calculated using eq. (34).
[h]Values are calculated using eq. (38).
[i]Values are calculated using eq. (44) and (48).

It is evident that the European option price and the pure option price provide lower and upper bound, respectively, for the American option price. As noted by BW (p. 318), the quadratic approximation works fairly well with *less than one year to expiration;* the quadratic approximation results, however, weaken considerably for longer maturity options. This is confirmed by the $T = 3.00$ figures in Tables I and II.

Looking at the new estimates, all four of them provide almost the same level of accuracy as the quadratic approximation for short maturity options. Considering $T = 3.00$, all four new methods provide better approximation than the quadratic for at-the-money and out-of-the-money options. While the quadratic estimate is better than MA, MB, and MC for in-the-money options, the MR estimate provides much better approximation than the quadratic.

Overall Simulation Results

The options in Tables I and II are only a few of the total number of options considered. Table III provides some summary statistics on the approximation or pricing error of the estimates considering all options (3121 calls, 3141 puts) in this study. Panel A concerns the call option pricing errors, while Panel B concerns the put option pricing errors.

Two measures of pricing error are used:

Dollar Error (DE): Price Estimate − Binomial Price

Absolute Percentage Error (APE): 100 × Absolute (DE)/ Binomial Price

The DE of the European option price is always non-positive and represents the early exercise premium of an American option. The corresponding APE expresses the early exercise premium as a percentage of the American price. For call (put) options, the average early exercise premium of 0.44 (0.45) in dollars and is 3.04% (3.20%) of the American price. As indicated by the standard deviation figures, the early exercise premium, however, varies widely across options. The DE of the pure option price is always non-negative and represents the expected interest savings of a pure option (compared to its American counterpart) due to its futures-style margining. The corresponding APE expresses the expected interest savings as a percentage of the American price. For call (put) options, the average expected interest savings is 0.75 (0.72) in dollars and is 7.11% (6.90%) of the American price. As indicated by the standard deviation figures, the expected interest savings of a pure option, however, varies widely across options.

TABLE III Summary Statistics on American Futures Option Pricing Errors of Quadratic Approximation and New Methods, MA, MB, MC, and MR

A. *Call Option Pricing Errors"*

(3121 Options, Binomial Price: Mean: 10.34, Std. Dev.: 7.82)

				New Methods			
	European	*Pure*	*Quadratic*	MA	MB	MC	*MR*
Dollar error:							
Mean	-0.44	0.75	0.09	-0.12	-0.13	-0.09	-0.00
Std. dev.	0.71	1.01	0.15	0.28	0.30	0.26	0.07
Absolute percentage error:							
Mean	3.04	7.11	1.42	0.87	0.93	0.76	0.36
Std. dev.	3.57	6.51	4.73	1.49	1.57	1.36	0.43
Percentage of cases where absolute percentage error is less than:							
5	81.10	49.63	95.32	96.64	96.28	97.21	99.97
2	53.57	17.05	80.Q1	86.25	85.29	88.40	98.78
1	32.20	5.51	63.06	75.59	73.89	79.56	93.66

B. *Put Option Pricing Errors"*

(3141 Options, Binomial Price: Mean: 10.29, Std. Dev.: 7.19)

				New Methods			
	European	*Pure*	*Quadratic*	MA	MB	MC	*MR*
Dollar error:							
Mean	-0.45	0.72	0.09	-0.05	-0.06	0.01	0.02
Std. dev.	0.71	0.96	0.15	0.15	0.17	0.10	0.06
Absolute percentage error:							
Mean	3.20	6.90	1.16	0.48	0.53	0.43	0.40
Std. dev.	3.77	6.44	1.73	1.00	1.08	0.75	0.67
Percentage of cases where absolute percentage error is less than:							
5	79.97	51.00	95.99	99.08	98.89	99.75	99.84
2	51.93	18.78	81.25	94.08	92.90	96.98	99.11
1	30.92	7.23	64.31	86.92	84.94	92.33	92.30

"The true price of an American futures call or put option is computed using the binomial method with 750 time steps. Alternative estimates of this call or put price are computed using the European formula [eq. (3) or (4)], the pure option formula [eq. (9) or (10)], the quadratic approximation method [eq. (7) or (8)], and the new methods MA [eq. (26) or (27)], MB [eq. (33) or (34)], MC [eq. (37) or (38)], and MR [eqs. (43) and (47), or eqs. (44) and (48)].

Dollar error of an estimate is calculated by subtracting the true price from the estimated price. The absolute percentage error is then calculated as 100 times the ratio of the absolute dollar error to the true price.

Alternative parameter values used in generating the true price and the estimates are: risk-free rate *(r)* = 0.04, 0.06, 0.08,0.10,0.12,0.14; volatility of futures price change relative (5) = 0.10, 0.13, 0.15,0.20,0.23,0.25,0.30,0.35,0.40; option's time to maturity (T) = 0.25,0.35,0.50,0.75,0.85, 1.00, 1.50, 2.00, 3.00; futures price *(F)* = 80, 85, 90, 95, 100,105,110,115,120.

The quadratic approximation method overestimates the American price by nine cents on average; the absolute approximation error is 1.42% (1.16%) of the American call (put) option price. As discussed earlier and is indicated by the standard deviations of DE and APE, the accuracy of the quadratic approximation varies across options.

All four methods (MA, MB, MC, MR) underestimate the American call price on average, although the average DE of the MR method is zero when rounded to two decimal places. The average APE of the four new call option estimates are all less than 1%, with the lowest average APE of 0.36% for the MR estimate.

The new methods MA and MB (MC and MR) underestimate (overestimate) the American put option price on average. The average APE of all four new estimates (MA: 0.48%, MB: 0.53%, MC: 0.43%, MR: 0.4%) is less than that of the quadratic estimate (1.16%). The standard deviation of all four new estimates' APE is also lower than that of the BW estimate. This is also true for call options. In addition to the averages and standard deviations, Table III reports the frequency with which the APE of a given approximation is less than 5%, 2%, and 1%. Higher frequency would mean *good* (*less than* 5% *APE*) approximation for a larger number (wider variety) of options.

Of the 3121 call options, the quadratic APE is less than 5% in 95.32% cases, less than 2% in 80.01% cases, and less than 1% in 63.06% cases. All of the corresponding frequencies are higher for the new methods. The differential between the quadratic estimate and the new estimates is most clear and substantial when the frequency of less than 1% APE is considered. Even the simplest new method, MA, provides a price estimate within 1% of the true price in more than three-quarters of the 3141 cases. The MR estimate virtually guarantees approximation error of less than 2%, irrespective of the option parameters. In about 94% of the cases, it provides a call price estimate within 1% of the true price.

Considering the 3141 put options, the quadratic APE frequencies remain similar as in the case of call options. The new methods provide good approximation for an even larger number (wider variety) of options. A less than 5% APE is virtually guaranteed no matter which of the four new methods is employed. Even the simplest method, MA, provides a price estimate within 1% of the true price of about 87% of the 3141 put options in this simulation. In comparison, the corresponding frequency is only 64% for the quadratic method.

Simulation Results by Call Option Features

The aggregate statistics discussed above do not reveal whether the accuracy of a given method or a lack thereof is uniform across options differing in moneyness, time to maturity, and volatility of the futures price change relative and different levels of the risk-free rate. The standard deviation of the dollar and absolute percentage error of the estimation methods (Table III) is symptomatic of varying accuracy of the methods across option features. An examination of the pricing error statistics by option features should help discern whether a particular estimation method is the most accurate across all option features; whether none of the methods is uniformly superior (in terms of accuracy); and, under what circumstances a given method is preferable.

The relevant statistics for the European option and the pure option estimates are included for the sake of completeness, but discussion focuses on the quadratic and the four new methods (MA, MB, MC, MR).

Call Option Simulation Results By Moneyness

Table IV presents summary information about approximation accuracy of the seven estimation methods (European, pure, quadratic, MA, MB, MC, MR) by five degrees of moneyness, viz., *deep-out-of-the-money* ($F = 80, 85$), *out-of-the-money* ($F = 90$), *at-the-money* ($F = 95, 100, 105$), *in-the-money* ($F = 110$), and *deep-in-the-money* ($F = 115, 120$). Included therein are the averages for the dollar error (DE) and absolute percentage error (APE) as well as the frequency with which a given method produces a price estimate within 5%, 2%, and 1% of the true American call option price (computed using the binomial method).

The new methods (MA, MB, and MC) produce more accurate price estimates than the quadratic approximation method for deep-out-of-the-money, out-of-the-money, and at-the-money call options.[24] The situation reverses for in-the-money and deep-in-the-money call options. The MR method is, however, the most accurate for any of the five degrees of moneyness. This is the only method which guarantees a price estimate within 5% of the true price for any call option.

For in-the-money and deep-in-the-money call options where the quadratic method performs its best, its accuracy is within 1% of the true price for 82% (89%) of the 351 (702) in-the-money (deep-in-the-money) call options. The much simpler MR method does even better

[24]For deep-out-of-the-money call options, the accuracy of the quadratic method is poorer than that of the European price.

TABLE IV

Summary Statistics on American Futures Call Option Pricing Errors of Quadratic Approximation and New Methods, MA, MB, MC, and MR: By Option Moneyness[a] [Exercise Price (X) = 100]

A. Deep-Out-of-the-Money (F = 80, 85) Call Options
(664 Options, Mean Binomial Option Price: 2.87)

	European	Pure	Quadratic	New Methods MA	MB	MC	MR
Mean dollar error	−0.09	0.35	0.08	−0.04	−0.04	−0.04	−0.02
Mean absolute % error	1.69	9.06	3.48	0.82	0.86	0.82	0.46
Percentage of cases where absolute percentage error is less than:							
5	90.66	37.20	82.83	97.59	97.44	97.59	99.85
2	71.54	7.98	54.97	87.20	86.30	87.20	96.99
1	51.21	0.90	34.34	73.50	72.29	73.65	89.61

B. Out-of-the-Money (F = 90) Call Options
(351 Options, Mean Binomial Option Price: 4.46)

	European	Pure	Quadratic	New Methods MA	MB	MC	MR
Mean dollar error	−0.14	0.48	0.09	−0.06	−0.06	−0.06	−0.01
Mean absolute % error	2.05	8.20	1.72	0.75	0.81	0.73	0.38
Percentage of cases where absolute percentage error is less than:							
5	88.61	43.31	93.16	97.44	97.15	97.44	100.00
2	66.95	10.83	68.95	87.75	86.90	88.03	98.86
1	44.16	1.14	43.88	78.63	76.92	78.92	91.17

C. At-the-Money (F = 95, 100, 105) Call Options
(1053 Options, Mean Binomial Option Price: 8.68)

	European	Pure	Quadratic	New Methods MA	MB	MC	MR
Mean dollar error	−0.31	0.75	0.10	−0.09	−0.10	−0.08	0.01
Mean absolute % error	2.78	7.30	1.02	0.78	0.85	0.71	0.39
Percentage of cases where absolute percentage error is less than:							
5	83.29	48.34	99.24	96.87	96.58	97.34	100.00
2	56.22	13.30	83.48	87.65	86.61	89.17	99.15
1	34.47	1.90	63.91	79.20	77.59	81.77	93.35

(continued)

TABLE IV (Continued)

D. In-the-Money ($F = 110$) Call Options
(351 Options, Mean Binomial Option Price: 14.75)

	European	Pure	Quadratic	New Methods MA	MB	MC	MR
Mean dollar error	−0.60	1.01	0.09	−0.17	−0.18	−0.11	0.02
Mean absolute % error	3.83	6.08	0.54	1.02	1.11	0.81	0.29
Percentage of cases where absolute percentage error is less than:							
5	75.78	55.84	100.00	95.73	96.16	96.30	100.00
2	41.88	21.08	93.73	83.76	82.34	88.32	99.72
1	19.09	6.55	82.05	73.22	71.80	80.91	98.86

E. Deep-in-the-Money ($F = 115, 120$) Call Options
(702 Options, Mean Binomial Option Price: 20.62)

	European	Pure	Quadratic	New Methods MA	MB	MC	MR
Mean dollar error	−1.02	1.12	0.08	−0.23	−0.25	−0.15	−0.01
Mean absolute % error	4.80	4.94	0.35	1.03	1.10	0.76	0.24
Percentage of cases where absolute percentage error is less than:							
5	67.66	63.39	100.00	95.44	94.87	97.01	100.00
2	31.77	32.34	97.15	83.76	83.05	88.60	99.43
1	11.40	16.95	89.03	71.80	69.37	81.48	96.58

[a]The true price of an American futures call option is computed using the binomial method with 750 time steps. Alternative estimates of this price are computed using the European formula [eq. (3)], the pure option formula [eq. (9)], the quadratic approximation method [eq. (7)], and the new methods MA [eq. (26)], MB [eq. (33)], MC [eq. (37)], and MR [eqs. (43) and (47)].

Dollar error of an estimate is calculated by subtracting the true price from the estimated price. The absolute percentage error is then calculated as 100 times the ratio of the absolute dollar error to the true price.

Alternative parameter values used in generating the true price and the estimates are: risk-free rate (r) = 0.04, 0.06, 0.08, 0.10, 0.12, 0.14; volatility of futures price change relative (s) = 0.10, 0.13, 0.15, 0.20, 0.23, 0.25, 0.30, 0.35, 0.40; option's time to maturity (T) = 0.25, 0.35, 0.50, 0.75, 0.85, 1.00, 1.50, 2.00, 3.00.

with a corresponding frequency of 99% (97%). In dollar terms, the MR method is on average within a penny (deep-in-the-money) or two (in-the-money) of the binomial price which averages 20.62 (14.75) for deep-in-the-money (in-the-money) call options.

Among the five degrees of moneyness, the most commonly traded are the at-the-money options.[25] For these options, while the MR method

[25]The definition of the at-the-money call option category used in this study includes the exactly at-the-money ($F = 100$) as well as the slightly out-of-the-money ($F = 95$) and slightly in-the-money ($F = 105$) call options.

is the most accurate, the MA method provides accuracy comparable to MB and MC methods, and better than the quadratic approximation method. Since it is the easiest to calculate, it may be used as an alternative to the MR choice.

Call Option Simulation Results by Time to Maturity

Table V presents summary information about approximation accuracy of the seven estimation methods (European, pure, quadratic, MA, MB, MC, MR) by three classes of maturity, viz., *short* ($T = 0.25, 0.35, 0.50$), *medium* ($T = 0.75, 0.85, 1.00$), and *long* ($T = 1.50, 2.00, 3.00$).

The MR method is the only one which virtually guarantees a price estimate within 5% of the binomial price. Irrespective of the option's maturity, the MR method yields a price estimate within a penny or two of the true price. The MA, MB, and MC methods provide better accuracy than the quadratic approximation method for all three maturity classes.

The most actively traded futures options are the short maturity ones. For these options, the easiest to implement MA method provides a very high level of accuracy. The average DE of the MA price estimates is zero, the average APE is 0.10% (quadratic: 0.92%) and it effectively guarantees (quadratic frequency: 91%) a price estimate within 1% of the binomial price. The MA method's accuracy is also excellent for option's maturity up to one year. The accuracy of the MB, MC, and MR methods is virtually the same as that of the MA method for options with less than a year to maturity.

It was noted in the previous section (refer to Table IV) that the MA, MB, and MC methods are not as accurate when the option is in-the-money or deep-in-the-money. The results in this section (refer to Panels A and B of Table V) suggest that so long as the option is maturing in less than a year's time these methods are quite accurate regardless of the option's degree of moneyness, and for that matter any other feature.

As noted by BW, the quadratic method's accuracy weakens significantly for long maturity options. So does the accuracy of the methods MA, MB, and MC, although the accuracy of these methods remains marginally better than that of the quadratic's. This is visible in Panel C of Table V. The MR method's accuracy is noticeably better than that of the others for long maturity call options. While the average DE (APE) of the quadratic approximation method is 0.22 (2.40%), the MR method, on average, produces a price estimate within a couple of pennies (0.56%) of

TABLE V

Summary Statistics on American Futures Call Option Pricing Errors of Quadratic Approximation and New Methods, MA, MB, MC, and MR: By Option Maturity[a] [Exercise Price $(X) = 100$]

				New Methods			
	European	*Pure*	*Quadratic*	*MA*	*MB*	*MC*	*MR*
A. Short maturity ($T = 0.25, 0.35, 0.50$) call options (1015 options, mean binomial option price: 8.08)							
Mean dollar error	−0.11	0.15	0.01	−0.00	−0.01	0.01	0.01
Mean absolute % error	0.89	2.32	0.92	0.10	0.11	0.14	0.22
Percentage of cases where absolute percentage error is less than:							
5	99.31	93.40	98.72	99.90	99.90	99.90	100.00
2	88.87	46.80	96.26	99.70	99.70	99.70	99.61
1	67.49	14.98	90.94	99.31	99.21	98.31	98.82
B. Medium maturity ($T = 0.75, 0.85, 1.00$) call options (1053 options, mean binomial option price: 9.84)							
Mean dollar error	−0.28	0.47	0.04	−0.04	−0.04	−0.01	0.01
Mean absolute % error	2.11	5.50	0.92	0.36	0.41	0.27	0.29
Percentage of cases where absolute percentage error is less than:							
5	93.07	49.00	98.01	100.00	100.00	100.00	100.00
2	58.41	5.22	88.60	99.05	98.77	99.53	99.62
1	27.73	1.90	69.99	92.21	89.27	96.87	97.53
C. Long maturity ($T = 1.50, 2.00, 3.00$) call options (1053 options, mean binomial option price: 13.02)							
Mean dollar error	−0.91	1.60	0.22	−0.31	−0.33	−0.26	−0.02
Mean absolute % error	6.04	13.34	2.40	2.12	2.25	1.85	0.56
Percentage of cases where absolute percentage error is less than:							
5	51.57	8.07	89.36	90.12	89.08	91.83	99.91
2	14.72	0.19	55.75	60.50	57.93	66.38	97.15
1	2.66	0.00	29.25	36.09	34.09	43.21	84.81

[a]The true price of an American futures call option is computed using the binomial method with 750 time steps. Alternative estimates of this price are computed using the European formula [eq. (3)], the pure option formula [eq. (9)], the quadratic approximation method [eq. (7)], and the new methods MA [eq. (26)], MB [eq. (33)], MC [eq. (37)], and MR [eqs. (43) and (47)].

Dollar error of an estimate is calculated by subtracting the true price from the estimated price. The absolute percentage error is then calculated as 100 times the ratio of the absolute dollar error to the true price.

Alternative parameter values used in generating the true price and the estimates are: risk-free rate $(r) = 0.04, 0.06, 0.08, 0.10, 0.12, 0.14$; volatility of futures price change relative $(s) = 0.10, 0.13, 0.15, 0.20, 0.23, 0.25, 0.30, 0.35, 0.40$; current futures price $(F) = 80, 85, 90, 95, 100, 105, 110, 115, 120$.

the true price (average: 13.02). The MR (quadratic) method's accuracy is within 1% for 85% (29%) of the 1053 long maturity call options. The MR method also effectively guarantees an approximation error of less than 5%.

Call Option Simulation Results by Volatility of Futures Price Change Relative and Interest Rate

Options are currently available on a wide range of underlying futures contracts (stock index, treasury, currency, commodities). The volatility of the underlying futures contract's price change relative thus varies widely. Also, the sensitivity of the option price to the risk-free interest rate is expected to vary with the nature of the underlying futures contract.

The approximation accuracy of the seven estimation methods (European, pure, quadratic, MA, MB, MC, MR) are examined by three ranges of volatility, viz., *low* ($s = 0.10, 0.13, 0.15$), *medium* ($s = 0.20, 0.23, 0.25$) and *high* ($s = 0.30, 0.35, 0.40$) and under three risk-free rate situations, viz., *low* ($r = 0.04, 0.06$), *moderate* ($r = 0.08, 0.10$) and *high* ($r = 0.12, 0.14$).[26] There is no noticeable variation in the accuracy of the estimation methods across different volatility situations.[27] The new methods (MA, MB, MC, and MR) provide better accuracy than the quadratic approximation under all three volatility situations. The accuracy of the quadratic and new methods (MA, MB, MC, and MR) worsens somewhat with a higher risk-free rate. Considering the average absolute percentage error and its frequency, the loss of accuracy is the most (least) with the quadratic (MR) method. All four new methods perform as well as, and in some cases better, than the quadratic approximation under all three interest rate situations.

Put Option Simulation Results

Tables VI and VII contain put option simulation results, respectively, by moneyness and time to maturity. Simulation results for different volatility and risk-free rate ranges are not reported here.[28]

These results are essentially similar to the ones discussed above for the call options. Between call and put options, the aggregate statistics in Table III as well as those in Tables IV through VII indicate that the new methods are even more accurate for put options. The quadratic approximation works roughly the same for both call and put options.

[26]A table of summary information is available upon request from the author.
[27]The quadratic and MR (MA, MB, and MC) method's accuracy improves (worsens) marginally for more volatile futures prices.
[28]These results are available upon request from the author.

TABLE VI

Summary Statistics on American Futures Put Option Pricing Errors of Quadratic Approximation and New Methods, MA, MB, MC, and MR: By Option Moneyness[a][Exercise Price $(X) = 100$]

A. Deep-Out-of-the-Money ($F = 115, 120$) Put Options
(684 Options, Mean Binomial Option Price: 3.90)

				New Methods			
	European	*Pure*	*Quadratic*	*MA*	*MB*	*MC*	*MR*
Mean dollar error	−0.13	0.45	0.10	−0.01	−0.02	−0.01	−0.01
Mean absolute % error	1.84	8.75	2.39	0.39	0.44	0.38	0.41
Percentage of cases where absolute percentage error is less than:							
5	89.91	38.45	85.67	99.42	99.42	99.42	99.42
2	70.18	8.63	58.19	95.61	95.03	96.20	98.10
1	49.56	0.59	35.97	88.74	86.55	89.62	91.67

B. Out-of-the-Money ($F = 110$) Put Options
(351 Options, Mean Binomial Option Price: 5.22)

				New Methods			
	European	*Pure*	*Quadratic*	*MA*	*MB*	*MC*	*MR*
Mean dollar error	−0.17	0.55	0.10	−0.01	−0.02	−0.00	−0.00
Mean absolute % error	2.13	8.16	1.65	0.36	0.41	0.34	0.38
Percentage of cases where absolute percentage error is less than:							
5	88.32	42.17	94.59	99.72	99.72	99.72	99.72
2	66.10	10.26	70.37	96.30	94.87	97.15	98.86
1	44.73	0.86	45.87	90.31	87.75	91.74	93.45

C. At-the-Money ($F = 105, 100, 95$) Put Options
(1053 Options, Mean Binomial Option Price: 8.68)

				New Methods			
	European	*Pure*	*Quadratic*	*MA*	*MB*	*MC*	*MR*
Mean dollar error	−0.31	0.75	0.10	−0.02	−0.03	0.02	0.03
Mean absolute % error	2.81	7.28	1.01	0.41	0.45	0.43	0.52
Percentage of cases where absolute percentage error is less than:							
5	83.00	48.91	99.15	99.81	99.53	100.00	100.00
2	55.84	13.30	84.33	95.54	94.21	97.82	98.96
1	33.33	1.99	64.77	90.31	88.32	94.59	86.80

continued

TABLE VI (Continued)

D. In-the-Money ($F = 90$) Put Options
(351 Options, Mean Binomial Option Price: 14.01)

	European	Pure	Quadratic	New Methods MA	MB	MC	MR
Mean dollar error	−0.59	0.93	0.08	−0.07	−0.09	0.02	0.04
Mean absolute % error	3.97	5.91	0.50	0.65	0.72	0.56	0.39
Percentage of cases where absolute percentage error is less than:							
5	74.64	56.98	100.00	97.72	97.44	99.72	100.00
2	40.17	22.51	94.30	90.60	89.46	96.30	100.00
1	17.38	8.83	82.91	81.77	79.77	92.59	97.15

E. Deep-in-the-Money ($F = 85, 80$) Put Options
(702 Options, Mean Binomial Option Price: 19.61)

	European	Pure	Quadratic	New Methods MA	MB	MC	MR
Mean dollar error	−1.05	0.93	0.06	−0.12	−0.14	0.01	0.01
Mean absolute % error	5.25	4.41	0.28	0.64	0.71	0.44	0.22
Percentage of cases where absolute percentage error is less than:							
5	64.25	67.81	100.00	98.01	97.72	99.72	100.00
2	27.07	39.32	98.01	91.03	89.60	96.72	100.00
1	8.97	23.93	91.17	80.91	79.49	91.74	98.15

[a]The true price of an American futures put option is computed using the binomial method with 750 time steps. Alternative estimates of this price are computed using the European formula [eq. (4)], the pure option formula [eq. (10)], the quadratic approximation method [eq. (8)], and the new methods MA [eq. (27)], MB [eq. (34)], MC [eq. (38)], and MR [eqs. (44) and (48)].

Dollar error of an estimate is calculated by subtracting the true price from the estimated price. The absolute percentage error is then calculated as 100 times the ratio of the absolute dollar error to the true price.

Alternative parameter values used in generating the true price and the estimates are: risk-free rate (r) = 0.04, 0.06, 0.08, 0.10, 0.12, 0.14; volatility of futures price change relative (s) = 0.10, 0.13, 0.15, 0.20, 0.23, 0.25, 0.30, 0.35, 0.40; option's time to maturity (T) = 0.25, 0.35, 0.50, 0.75, 0.85, 1.00, 1.50, 2.00, 3.00.

Early Exercise Results

The new methods (MA, MB, MC, MR) do not use the critical futures prices as an input for valuing American futures options. Therefore, the simulation cases where early exercise is optimal according to the quadratic approximation method are examined more closely. The critical futures price is calculated for all 3121 call options and 3141 put options using the quadratic approximation method.[29] Early exercise of

[29]The iterative procedure is described in BW.

TABLE VII

Summary Statistics on American Futures Put Option Pricing Errors of Quadratic Approximation and New Methods, MA, MB, MC, and MR: By Option Maturity[a] [Exercise Price (X) = 100]

				New Methods			
	European	*Pure*	*Quadratic*	*MA*	*MB*	*MC*	*MR*
A. Short maturity (T = 0.25, 0.35, 0.50) put options (1035 options, mean binomial option price: 7.93)							
Mean dollar error	−0.12	0.14	0.01	0.00	−0.00	0.02	0.01
Mean absolute % error	0.99	2.26	0.38	0.17	0.16	0.23	0.28
Percentage of cases where absolute percentage error is less than:							
5	98.84	93.62	98.74	99.61	99.61	99.61	99.71
2	86.28	48.50	97.01	99.32	99.32	99.32	99.32
1	65.70	17.68	92.27	98.84	98.84	98.84	97.87
B. Medium maturity (T = 0.75, 0.85, 1.00) put options (1053 options, mean binomial option price: 9.86)							
Mean dollar error	−0.29	0.45	0.04	−0.01	−0.02	0.03	0.03
Mean absolute % error	2.28	5.34	0.88	0.23	0.25	0.30	0.39
Percentage of cases where absolute percentage error is less than:							
5	91.45	50.14	98.10	99.91	99.91	99.91	99.81
2	56.60	7.22	89.55	99.72	99.62	99.81	99.72
1	25.64	3.61	69.99	96.11	94.97	99.05	93.07
C. Long maturity (T = 1.50, 2.00, 3.00) put options (1053 options, mean binomial option price: 13.04)							
Mean dollar error	−0.93	1.58	0.22	−0.13	−0.15	−0.02	0.01
Mean absolute % error	6.29	13.03	2.22	1.04	1.18	0.75	0.51
Percentage of cases where absolute percentage error is less than:							
5	49.95	9.97	91.17	97.72	97.15	99.72	100.00
2	13.49	1.14	57.46	83.29	79.87	91.83	98.29
1	1.99	0.57	31.15	66.00	61.25	79.11	86.04

[a]The true price of an American futures put option is computed using the binomial method with 750 time steps. Alternative estimates of this price are computed using the European formula [eq. (4)], the pure option formula [eq. (10)], the quadratic approximation method [eq. (8)], and the new methods MA [eq. (27)], MB [eq. (34)], MC [eq. (38)], and MR [eqs. (44) and (48)].

Dollar error of an estimate is calculated by subtracting the true price from the estimated price. The absolute percentage error is then calculated as 100 times the ratio of the absolute dollar error to the true price.

Alternative parameter values used in generating the true price and the estimates are: risk-free rate (r) = 0.04, 0.06, 0.08, 0.10, 0.12, 0.14; volatility of futures price change relative (s) = 0.10, 0.13, 0.15, 0.20, 0.23, 0.25, 0.30, 0.35, 0.40; futures price (F) = 80, 85, 90, 95, 100, 105, 110, 115, 120.

a call (put) option is deemed to be optimal if the critical futures price F^* (F^{**}) is *below* (*above*) the current futures price, F.

Ninety-one call and 151 put options are identified for which the quadratic approximation method predicts an optimal early exercise. It is found that the quadratic and MA, MB, MC, and MR price estimates are all equal to the binomial price which is, in turn, equal to the intrinsic value of the option.

SUMMARY AND CONCLUSIONS

An American pure option (futures option with futures-style margining) has the same value as a European pure option [(Chen and Scott (1993); Lieu (1990)]. Further, the value of a conventional (stock-style margining) American futures option is bounded above by the pure option value [Chaudhury and Wei (1994)]. Exploiting these results, this article demonstrates that the American futures option value (early exercise premium) may be calculated as a multiple (proportion) of the European futures option value. Traditionally [Barone-Adesi and Whaley (1987); Kim (1994)], the American futures option value is calculated as the sum of the European futures option value [Black (1976)] and the early exercise premium.

One insight derived from the proposed *multiplicative* valuation framework is that marking-to-market (futures-style margining) a European futures option is more valuable than either adding the early exercise privilege or marking-to-market an American futures option. The early exercise feature serves the same purpose (viz., freeing up capital) as the marking-to-market mechanism, but only in situations where early exercise is imminent.

Depending on how the multiple that applies to Black's (1976) European futures option value is estimated, four new methods (MA, MB, MC, and MR) of calculating American futures option prices are offered in this article. They share the closed form nature of Black's formula but avoid iterative search for the critical futures price (the quadratic approximation method of Barone-Adesi and Whaley, 1987) or numerical integration [Kim (1994)]. Hence, they are as easy to implement as Black's formula on a hand-held calculator or a spreadsheet. As shown by our simulation results, their accuracy is better than Black's formula.

The main simulation results are summarized in Table VIII. For options which are in-the-money by 10% or more (in-the-money and deep-in-the-money), either the MR method or the quadratic approximation should be used with preference for the MR method because it is easier to implement and still performs better. The most active futures option contracts are usually less than 10% away-from-the-money (at-the-money). For these as well as those options which are out-of-the-money by 10% or more (out-of-the-money and deep-out-of-the-money), the MR or MA method should be used.

TABLE VIII

Summary of American Futures Option Simulation Results for Quadratic Approximation and New Methods, MA, MB, MC, and MR

Option Feature	*Simulation Result*
Moneyness	All four new methods produce more accurate price estimates than the quadratic approximation for at-the-money, out-of-the-money and deep-out-of-the-money options.
	For in-the-money and deep-in-the-money options, the MR method and the quadratic approximation are more accurate than the MA, MB, and MC methods. The MR method performs relatively better than quadratic approximation. The MR method is, on average, within a penny or two of the binomial price.
Time to maturity	All four new methods produce more accurate price estimate than the quadratic approximation for any maturity.
	The simplest method, MA, provides excellent accuracy for options up to one year maturity; this does not depend on other option features (e.g., moneyness).
	The performance of all methods deteriorates for long maturity options. The quadratic approximation's accuracy weakens considerably. The MR estimate is still within a couple of pennies of the binomial price on average. The MR method guarantees a price estimate within 5% of the binomial price.
Futures price volatility	All four new methods produce more accurate price estimates than the quadratic approximation, irrespective of the level of futures price volatility.
Interest rate	All four new methods perform as well as quadratic approximation, and sometimes better than quadratic approximation, irrespective of the interest rate level.
	The accuracy of all methods worsens somewhat with higher interest rates; the loss of accuracy is the most (least) with the quadratic approximation (MR) method.

Much of the futures options trading volume is accounted for by options maturing in less than a year's time. For these options, the MA method is recommended since it is the simplest and still provides highly accurate option values. In the case of longer (than a year) maturity options, performance of the quadratic approximation weakens considerably. Although the accuracy of the new methods deteriorate as well, all four of them provide more accurate option values than the quadratic approximation. The MR method may still be used since it is within a couple of pennies of the true (binomial) price; it guarantees a pricing error of less than 5%. For added accuracy, Kim's (1994) method may be desirable.

Overall, the methods (specially MA and MR) presented in this article offer a set of attractive alternatives to practitioners and researchers who currently use Black's European formula. The new methods are more accurate and are as easy to use. Individuals without access to advanced computing equipment and/or lacking interest in the knowhow should find the new methods quite useful since they are no more complex than Black's formula. The proposed methods would be equally attractive to brokers and exchange officials who have traditionally taken recourse to Black's formula in calculating margin requirements.

An important use of the proposed methods can be made in ISD estimation, i.e., in assessing market participants' expectation of futures (and spot) price volatility. Since the true volatility is not known in real life, practitioners and researchers often use some form of weighted ISD in the valuation function to estimate model option prices. The resultant model prices are in turn compared to actual market prices to identify profitable trading opportunities and/or test market efficiency. Futures options based ISD's are used in evaluating the risk–return prospect of spot and futures positions and to calculate hedge ratios and other option sensitivities by agricultural producers, processors, operators, insurers, and policy makers to managers of currency, bond, stock, and swap portfolios. Other uses of ISD include setting margin requirements by exchanges, event studies dealing with the volatility spot and futures prices, investigation of the temporal behavior of futures and spot price volatility, and so forth.

To all these users of ISD, the proposed methods should be much more employable in a computational sense than the existing methods of American futures option valuation. The use of the proposed methods would be virtually similar to employing Black's formula, but the upward bias in ISD estimates from the latter would be avoided.

In light of the simulation results, the use of the MA method is recommended if ISD is estimated from the shorter (one year or less) maturity options, alone. If longer maturity options are included, the MR method should be used since it is the most consistent in terms of accuracy across all types of options. The quadratic approximation method performs poorly for longer maturity options. While Kim's (1994) method would perhaps yield more accurate ISD estimates than the MA or the MR method, the computational involvement would be significantly more. The computational advantage of using the MA or MR method over the quadratic approximation or Kim's (1994) method should be significant when a large number of option price observations such as transaction data are employed to form a single ISD estimate [e.g., Whaley (1986)].

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