UPPER BOUNDS FOR AMERICAN OPTIONS

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ABSTRACT

This paper provides a fuller characterization of the analytical upper bounds for American options than has been available to date. We establish properties required of analytical upper bounds without any direct reliance on the exercise boundary. A class of generalized European claims on the same underlying asset is then proposed as upper bounds. This set contains the existing closed form bounds of Margrabe (1978) and Chen and Yeh (2002) as special cases and allows randomization of the maturity payoff. Owing to the European nature of the bounds, across-strike arbitrage conditions on option prices seem to carry over to the bounds. Among other things, European option spreads may be viewed as ratio positions on the early exercise option. To tighten the upper bound, we propose a quasi-bound that holds as an upper bound for most situations of interest and seems to offer considerable improvement over the currently available closed form bounds. As an approximation, the discounted value of Chen and Yeh's (2002) bound holds some promise. We also discuss implications for parametric and nonparametric empirical option pricing. Sample option quotes for the European (XEO) and the American (OEX) options on the S&P 100 Index appear well behaved with respect to the upper bound properties but the bid–ask spreads are too wide to permit a synthetic short position in the early exercise option.

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1. INTRODUCTION

Analytical bounds for American option prices are interesting from both theoretical and practical perspectives. They provide theoretical restrictions for arbitrage-free pricing and optimal early exercise of American options. As most American option valuation problems require simultaneous determination of the early exercise boundary, their practical implementation involves numerical methods that may become computationally burdensome, in particular when there are multiple state variables.[1] Bounds, especially if they are analytical and closed form, can be useful in such circumstances in providing valuation guidelines,[2] developing approximations,[3] implying information from the observed American option prices,[4] setting trading restrictions such as dollar margin requirements on written options, managing the market risk of American option portfolios, and determining capital adequacy rules for institutional portfolios.

The aim of this paper is twofold. First, we specify some general properties of the analytical upper bounds for American options where the bounds themselves are construed as contingent claims on the same asset. Second, we propose as upper bounds a class of generalized European claims that are not specific to preferences and are also independent of the exercise boundary. Together they provide a rigorous economic characterization of the analytical upper bounds for American options that is intuitively appealing. At the same time these bounds are easier to compute and invert, and hence should be useful for valuation guidance and information extraction purposes.

There are three distinct and parallel lines of existing research on option bounds. Bounds based on the physical distribution or moments (e.g., Perrakis & Ryan, 1984; Lo, 1987; Grundy, 1991) are primarily limited to European options.[5] Bounds relying on the exercise policy/boundary of American options (e.g., Broadie & Detemple, 1996, Rogers, 2002, Andersen & Broadie, 2004) tend to produce tighter bounds. These bounds are most useful in situations where optimal exercise and accurate option valuation is the main focus (like executive and employee options) and/or the options do not have a liquid secondary market (like many OTC and structured products). However, typically these bounds are not in closed form even when they are analytic, i.e., they require iterative optimization or regression. Accordingly, their use is highly restrictive in dealing with large datasets and in the presence of multiple state variables, especially in implying information from the observed American option prices. For example, the vast majority of exchange-traded equity options are American options, and so are some widely popular index (e.g., S&P 100) or exchange traded fund (e.g., S&P 500

Depository Receipts known as SPIDERS, NASDAQ 100 Trust known as Cubes) options. Implying information from these option prices, using say a stochastic volatility model, can be a formidable task if one takes the early exercise boundary route.

The third type of bounds are neither preference-dependent, nor do they have any direct reliance on exercise policies/boundaries (e.g., Margrabe, 1978, Chen & Yeh, 2002, Chaudhury & Wei, 1994, Chung & Chang, 2005). Instead this line of bounds research looks for analytic functions in closed form to bound American options. Obviously these bounds may not be as suitable as the early exercise based bounds when the highest level of accuracy in valuation and exercise are of primary importance. However, the analytic closed form bounds can be quite useful in dealing with large datasets and especially in implying information from observed American option prices.

In prior research on closed form analytical bounds, Chaudhury and Wei (1994) and Melick and Thomas (1997) offer bounds for American futures options, but they do not apply to American put options on spot assets such as stocks, bonds, and foreign currency. Most recently, Chen and Yeh (2002) have provided closed form[6] upper bounds that are applicable to these options as well and are quite fast computationally.[7] Chung and Chang (2005) further generalize Chen and Yeh's bounds and extend the approach to the case of options on multiple assets.

Margrabe (1978) first noticed that the value of a European put option with the strike price compounded at the risk-free rate is an upper bound for the American put option. Chen and Yeh (2002)'s upper bound, on the other hand, is the expected maturity payoff or the pure (futures-style margining) option value of a European put option on a fraction of the asset, where the fraction adjusts for the net growth of the asset. We shall henceforth refer to these bounds as the Adjusted Strike European option (AKE) and the Adjusted Asset Pure European option (ASPE) bounds, respectively. The importance of these bounds is that they do not require the knowledge of the early exercise boundary and are as easy to calculate as the European option value.

This paper builds on prior works on analytical upper bounds for American spot options in several ways. First, while an impressive literature exists on the characterization of American option upper bounds in terms of the exercise boundary, an independent characterization of the analytical upper bounds themselves is lacking. An important objective of this paper is to fill this gap. We develop fundamental properties required of an analytical upper bound that is a contingent claim on the same underlying asset and share the same maturity as the American option. Since our characterization is

completely in terms of the value of claims and not their boundaries, this should enhance our understanding of the economic nature of American options and their bounds.

Second, we propose a set of generalized European claims as upper bounds for an American spot option that contains the AKE and ASPE bounds as special cases. An important benefit of a European claim as an upper bound is that its value is considerably easier to calculate than the target American option while the specific option valuation setup remains largely in tact. Since Chen and Yeh's (2002) bounds are pure option values or expected maturity payoffs and not European options, the bounding European options of this paper also help to tighten the bounds.

In a closely related work and citing an earlier version of this paper, Chung and Chang (2005) generalizes Chen and Yeh's bounds to analytic functions that amount to adjusting both the strike price and the units of assets of standard European options. As discussed later (Footnote 15), their generalized bounds are in fact special cases of the generalized European claims in this paper. Also, they do not provide full economic characterization of these bounds.

It is to be noted that the analytical bounds of Chen and Yeh (2002), Chaudhury and Wei (1994), and this paper are bounds on model option values under the hypothesized distribution. While there is no restriction on the nature of the distribution,[8] as mentioned earlier the closed form analytical bounds are more useful when there are multiple state variables, or when the exercise boundary of neither the target option nor the bounding claim is of interest. This is a special appeal of the European claims proposed in this paper as bounding claims.

Third, although the primary role of an upper bound is to provide a ceiling on the American option's value and guidance in regards to its exercise policy, other potentially important implications of an upper bound has not drawn much attention in the literature. In this paper, we discuss several of these implications. In the context of arbitrage conditions on option prices across various strikes, an interesting question is whether the respective upper bounds also satisfy similar conditions. This seems like a desirable property of a set of upper bounds as the credibility of the upper bounds in tracking the American options is enhanced. Another interesting implication of the generalized European claims of this paper as upper bounds is that European option spreads across different strikes essentially allow trading of early exercise options without ever trading the American options. Implications like this along with the traditional role of a price ceiling make upper bounds quite relevant for empirical option pricing.

Although a detailed empirical study is beyond the scope of this paper, we examine sample option quotes for the European (XEO) and the American (OEX) options on the S&P 100 Index. These quotes appear well behaved with respect to the upper bound properties. A synthetic long position in the early exercise option seems quite expensive and the bid–ask spreads are too wide to permit a synthetic short position in the early exercise option. This could explain why the XEO contracts are not as popular as the OEX contracts and would suggest redesigning the OEX contracts purely as early exercise options.

Lastly, despite their many benefits, one weakness of the analytical upper bounds that do not rely on the exercise boundary is that the bounds themselves are not quite accurate in approximating the American option value. To this end, we propose a quasi-bound that leads to significant improvement in pricing accuracy over the AKE and ASPE bounds. While the quasi-bound is truly an upper bound for most situations of interest, there still remains a set of circumstances where it is not meaningful.

To summarize, the contribution of this paper lies in providing a thorough economic characterization of the analytical upper bounds for American options using (generalized) European claims that are more tractable both intuitively and computationally. An analytical and closed form quasi-bound is also proposed that is tighter and covers most practical situations. Further, the paper discusses novel implications for empirical pricing of options, spreads, and the early exercise option.

As the dividend yield or leakage for most spot options is less than the risk-free rate, the pure option upper bound applies to American call options on these spot assets. Accordingly, we focus on spot put options in this paper. Section 2 specifies fundamental requirements for upper bounds on American options. In Section 3, a set of generalized and possibly randomized European claims (and a set of generalized but nonrandomized American claims) is proposed as upper bounds for the standard American options. We then discuss in Section 4 some interesting implications of our characterization of American option upper bounds. In section 5, we propose a quasi-bound that is truly an upper bound in most situations of interest. We present some numerical results to show the improvement in pricing accuracy offered by the quasi-bound. While an empirical study is beyond the scope of this paper, we briefly examine in Section 6 the applicability of bounds using sample CBOE option quotes for the S&P 100 European (XEO) and American (OEX) option contracts. Lastly, Section 7 summarizes and concludes the paper.

2. THE FUNDAMENTAL UPPER BOUND REQUIREMENTS

Let S_t denote the current price of the underlying spot asset with continuous and possibly stochastic leakage rate δ_t, and let V_t (v_t) stand for the current value of an American (European) option with strike price K and maturity time (time to maturity) T ($\tau = T-t$). The intrinsic value or immediate exercise proceeds, X_t, of V_t is $(K-S_t)^+$ for a put option and $(S_t-K)^+$ for a call option, where $(Y)^+$ indicates Max $(0, Y)$. The fact that the European option value v_t may fall below this intrinsic value X_t, and that the American option must satisfy the moving boundary condition $V_t \geq X_t$, is at the heart of the American option valuation problem.

In the rest of the paper, we assume that a risk-neutral or equivalent martingale measure exists and all expectations and moments are under this measure. The instantaneous risk-free rate at time t is r_t and it is allowed to vary stochastically over time unless mentioned otherwise. The discount factor for valuing at time t the j-period hence cash flows is thus $R_{t,j} = \exp(-\int_t^{t+j} r_s \mathrm{d}s)$.

Our objective here is to bind V_t from above by the value G_t of another contingent claim on the same asset and having the same maturity time T. Also, for obvious reason, we restrict our analysis to contingent claims with convex payoffs. Let us define such claims as the generalized claims, G. The intrinsic or immediate exercise value of G, if it is an American claim, is to be denoted as $X_{G,t}$.

The foremost requirement for an upper bound is stated in the following lemma.

Lemma 1. To qualify as an upper bound for the American option value V_t, the value G_t of the generalized claim must never fall below the intrinsic value X_t of V_t .

Proof: The American option value V_t is the greater of its continuation value and its immediate exercise or intrinsic value, where the continuation value itself represents the value of capturing potentially higher intrinsic value at a future time. Therefore, if G_t falls below X_t over any range of the asset price S_t, then G_t cannot be an upper bound for V_t. *QED*

The converse of Lemma 1 is an important result on bounding the American option.

Lemma 2. If the value G_t of the generalized claim never falls below the intrinsic value X_t of the American option, then G_t is an upper bound for V_t.

Proof: Starting at maturity time T, we have $V_T = X_T$ and $G_T = X_{G,T}$. At time $T-\Delta$, $V_{T-\Delta} = \text{Max}[R_{T-\Delta,\Delta}E_{T-\Delta}(V_T), X_{T-\Delta}]$, and $G_{T-\Delta} = R_{T-\Delta,\Delta}E_{T-\Delta}(G_T)$ if G is European and $G_{T-\Delta} = \text{Max}[R_{T-\Delta,\Delta}E_{T-\Delta}(G_T), X_{G,T-\Delta}]$ if it is American. Now, suppose $G_T \geq V_T$. Then, at time $T-\Delta$, the discounted value of the generalized claim is greater than the continuation value of the American option, i.e., $R_{T-\Delta,\Delta}E_{T-\Delta}(G_T) > R_{T-\Delta,\Delta}E_{T-\Delta}(X_T)$. If, in addition, $G_{T-\Delta} \geq X_{T-\Delta}$, then we have $G_{T-\Delta} \geq V_{T-\Delta}$, as $V_{T-\Delta} = \text{Max}[R_{T-\Delta,\Delta}E_{T-\Delta}(X_T), X_{T-\Delta}]$. Continuing backward, the discounted value of G would be no less than the continuation value of V, and if in addition $G_{T-2\Delta} \geq X_{T-2\Delta}$ prevails, then once again $G_{T-2\Delta} \geq V_{T-2\Delta}$, where $V_{T-2\Delta} = \text{Max}[R_{T-2\Delta,\Delta}E_{T-2\Delta}(V_{T-\Delta}), X_{T-2\Delta}]$. By continuing to work backward, it is shown that $G_t \geq V_t$. *QED*.

Lemma 2 and its proof closely follow the Theorem 1 of Chen and Yeh (2002) and their method of proof. However, there are some important differences here. Let us first reproduce their Theorem 1 below.

Theorem 1 of Chen and Yeh (2002):

> *"An American option is bounded from above by the risk neutral expectation of its maturity payoff if this expectation is greater than the intrinsic value at all times."*

This theorem is quite general and only assumes that the risk-neutral measure exists and the discount factor is strictly between 0 and 1 over all sample paths. In our terminology, the generalized claim G is an American claim in Chen and Yeh's Theorem 1. The risk-neutral expectation of the maturity payoff of G is in fact the price of a pure European option that is a European option with futures-style margining. If the value of a pure European option always exceeds its intrinsic value, then the early exercise feature does not add any value and as such the pure American option value equals the pure European option value. Since a pure American option is clearly more valuable than a conventional American option, the pure option value serves as an upper bound.[9]

For spot options, however, the pure European option value may fall below the intrinsic value. This may occur for call options if there is a high positive leakage on the underlying asset. For put options, just a low enough asset price may cause the pure European option value to fall below its intrinsic value. Therefore, the pure American option value exceeds the pure European option value (expected maturity payoff).[10] Although the pure American option value exceeds the conventional American option value, the

pure European option value or the expected maturity payoff is no longer an upper bound for the conventional American option value over the entire range of asset prices. In other words, if the claim G is an American call option with high leakage or a standard American spot put option, then the expected maturity payoff or the pure European value of G is not an upper bound for G.[11]

There are three critical ways our Lemma 2 improves upon Chen and Yeh's Theorem 1. *First*, Chen and Yeh's theorem concerns bounding the American option value by its own expected maturity payoff or pure European value. Lemma 2 here, on the other hand, uses a generalized claim G, possibly different, from the target American option V that we wish to bind from above. Thus our Lemma 2 enlarges the set of bounding claims compared to Chen and Yeh.[12] *Second*, Lemma 2 is general enough to allow generalized bounding claims that are European as long as the intrinsic value condition of Lemma 2 is met. This flexibility is due to the fact that G is possibly different from V, although both are contingent claims on the same spot asset and have same maturity. The simplest examples of G_t are S_t to bind the standard American call option and K to bind the standard American put option. Since European claims are less valuable than pure European claims (Expected Maturity Payoff), our Lemma 2 opens up a set of tighter upper bounds. *Third*, the American or early exercise feature is most valuable for the standard American put options. However, Chen and Yeh's Theorem 1 does not directly apply to standard American put options. Our Lemma 2, on the other hand, applies to all American put options including the standard ones.

It is, of course, possible to combine Chen and Yeh's Theorem 1 and our Lemma 2 to suggest the requirement for the expected maturity payoff or pure European value $E_t(X_{G,T})$ of a generalized American claim to be an upper bound for the standard American spot option value V_t.

Lemma 3. A standard American spot option's value V_t is bounded from above by $E_t(X_{G,T})$, the risk neutral expectation of the maturity payoff (or the pure European option value) of a generalized American option, if the generalized American option G satisfies the conditions: (a) $E_t(X_{G,T})$ is never less than its own intrinsic value $X_{G,t}$, and (b) G_t never falls below the intrinsic value X_t of V.

Proof: If condition (a) above is met, then by Chen and Yeh's Theorem 1, $E_t(X_{G,T})$ is an upper bound for the American option value G_t of the generalized claim. If condition (b) is satisfied, then by Lemma 2, G_t bounds V_t

from above. Therefore, combining (a) and (b), $E_t(X_{G,T})$ is an upper bound for V_t. *QED*.

The following corollary gives the sufficient condition for the condition (b) of Lemma 3 to hold.

Corollary 1. A sufficient condition for the generalized claim's American option value G_t to stay above the intrinsic value X_t of the standard American option V is that G's intrinsic value $X_{G,t}$ never falls below V's intrinsic value X_t.

Proof: If $X_{G,t}$ never falls below X_t, then the claim G dominates the claim V in terms of payoff under all circumstances. Hence, to prevent arbitrage, the price G_t must be at least as high as the price V_t . But by the intrinsic value boundary condition, $V_t \geq X_t$. Therefore, $G_t \geq X_t$ follows. *QED*.

It is important to note two things. *First*, Corollary 1 provides a condition that relates just the intrinsic values of the bound and the American option. To our knowledge, such a relationship is novel. *Second*, Lemma 3 and Corollary 1 apply to generalized American claims. According to Lemma 2, this American aspect in and of itself is not necessary to bound V_t. A good example of a European generalized claim that satisfies Lemma 2 and thus bounds the standard American option is Margrabe's (1978) AKE option. We shall shortly discuss such claims.

3. NEW GENERALIZED CLAIMS AS UPPER BOUNDS

We now turn to the important task of structuring a generalized claim G such that G_t satisfies the requirements for bounding the American spot option value V_t. We propose a general structure, not specific to any stochastic process for the underlying asset. Although it is not essential, we assume for convenience a constant interest rate r and a constant leakage rate δ.[13] The net risk-neutral drift of the asset is thus assumed to be a constant $y = (r-\delta)>0$.[14]

Lemma 4. Suppose G is a European put option with the following maturity payoff function: $X_{G,T} = [e^{r(T-t)}\varepsilon_T K - e^{\delta\,(T-t)}\eta_T S_T]^+$, where ε_T and η_T are positive random variables with $E_t(\varepsilon_T) = 1$, $E_t(\eta_T) = 1$, $Variance_t(\varepsilon_T) = \sigma_\varepsilon^2$, $Variance_t(\eta_T) = \sigma_\eta^2$, and $Covariance_t\ (\varepsilon_T, S_T) = Covariance_t\ (\eta_T, S_T) = Covariance_t(\varepsilon_T, \eta_T) = 0$, for all t under the risk-neutral measure. Then, the generalized European put option's value G_t is an upper bound for the standard American put option's value V_t.

Proof:

$$
\begin{aligned}
G_t &= e^{-r(T-t-)} E_t[e^{r(T-t)}\varepsilon_T K - e^{\delta(T-t)}\eta_T\, S_T]^+ \\
&\geq e^{-r(T-t)}[KE_t(e^{r(T-t)}\varepsilon_T) - E_t(e^{\delta(T-t)}\eta_T S_T)]^+ \quad \text{(by convexity)} \\
&= [K - e^{-(r-\delta)(T-t)} E_t(S_T)]^+ \quad \text{(by assumption)} \\
&= [K - S_t]^+ \quad \text{(given the drift of the asset)}
\end{aligned}
$$

Since $G_t \geq [K - S_t]^+ = X_t$, for all t, then by Lemma 2, $G_t \geq V_t$. *QED*.

The European put option G generalizes the conventional put option to a situation where the buyer at time t has the right to sell, at time T, a random number $e^{\delta(T-t)}\,\eta_T$ of asset units for a random total price of $e^{r(T-t)}\varepsilon_T K$. The upper bound G_t, on the other hand, is easy to compute once the risk-neutral distribution for the terminal asset price is specified.

The AKE bound is a special case of the generalized European put option here, with deterministic $\varepsilon_T = \eta_T = 1$. With constant interest and leakage rates, the ASPE bound for the American put option is: $G_{CY,t} = E_t(K - e^{-(r-\delta)(T-t)}ST)^+$; this bounding put option is a pure European option with $\varepsilon_T = \eta_T = e^{-r(T-t-)}$ Either compounding the strike price (without slicing the optioned amount of the asset), or slicing the optioned amount of asset (leaving the strike unchanged), essentially enhances moneyness of the bounding claim relative to V.[15] We shall revisit this interesting insight later in the paper.

The economic intuition behind the AKE and ASPE bounds and the generalized European claims here is that by enhancing the moneyness of the option, these bounds are effectively factoring in the present value of expected net interest earnings in the exercise region.[16] The value of these bounds never falls below the intrinsic value of the standard American option and as such the implicit boundary is the strike price K of the standard American option. The strike price K is of course higher than the actual time varying exercise boundary of the standard American option. Consequently, the upper bounds reflect a higher present value of expected net earnings in the extended exercise region and end up bounding the standard American option's value. The potential role that we have in mind for the randomization of the maturity payoff is to make the lock-in value (notional early exercise value) of the generalized European option uncertain although maintaining the same expected value. More work is needed to fully explore the implications of the randomization scheme.

Note that, in AKE and ASPE and in Lemma 4, a different generalized European option bounds the American option value at different points of the latter's life. This can be realized from the presence of the time to

maturity $(T-t)$ in the maturity payoff. In contrast, the time to maturity does not appear in the maturity payoff function for standard European and American options. The bounding G at time t corresponds to a European option to sell $e^{\delta\,(T-t)}\eta_T$ units of asset for a total price of $e^{r(T-t)}\varepsilon_T K$ at time T. But the bounding G at time $t+j$ corresponds to a European option to sell $e^{\delta\,(T-t-j)}\eta_T$ units of asset for a total price of $e^{r(T-t-j)}\varepsilon_T K$ at time T.

In other words, to bound future values of the standard American put option, the bounding option G would call for selling rights on fewer units of the asset at a lower total price at maturity. As maturity approaches, the bounding G's optioned number of asset units approach the still unknown random number η_T and the total exercise price tends to K times the still unknown random number ε_T. Thus, once the random numbers realize at maturity, the payoff of Lemma 4's G may not bound V's payoff. This terminal weakness of Lemma 4's European G arises because it never has a meaningful intrinsic or exercise value of its own as the proposed randomization leaves the number of optioned asset units and the total exercise price undetermined until at maturity. Under stochastic interest and leakage rates, a similar situation arises with the AKE and ASPE bounds. While the expected strike price at maturity (AKE) and the expected number of optioned asset units (ASPE) are always known as they are in Lemma 4 here, the exact numbers are known only at maturity.

Interestingly, perhaps not unexpectedly, the G that bounds V_t, continues to bound the future values of V until exactly at maturity. Let us denote the value of t-specific bounding claim G's value at time $t+j$ as $G_{t,t+j}$.

Corollary 2. Suppose G is a European put option with the following maturity payoff function: $X_{G,T} = [e^{r(T-t)}\varepsilon_T K - e^{\delta(T-t)}\eta_T S_T]^+$, where ε_T and η_T are positive random variables with $E_t(\varepsilon_T) = 1$, $E_t(\eta_T) = 1$, $Variance_t(\varepsilon_T) = \sigma_\varepsilon^2$, $Variance_t(\eta_T) = \sigma_\eta^2$, and $Covariance_t(\varepsilon_T, S_T) = Covariance_t(\eta_T, S_T) = Covariance_t(\varepsilon_T, \eta_T) = 0$, for all t under the risk-neutral measure. Then, the generalized European put option's value $G_{t,t+j}$ is an upper bound for the standard American put option's value V_{t+j} for $0 \leq j < T$.

Proof:

$$
\begin{aligned}
G_{t,t+j} &= e^{-r(T-t-j)} E_{t+j}[e^{r(T-t)}\varepsilon_T K - e^{\delta(T-t)}\eta_T S_T]^+ \\
&\geq e^{-r(T-t-j)}[K E_{t+j}(e^{r(T-t)}\varepsilon_T) - E_{t+j}(e^{\delta(T-t)}\eta_T S_T)]^+ \quad \text{(by convexity)} \\
&= [K e^{rj} - e^{-(r-\delta)(T-t)} E_{t+j}(S_T) e^{rj}]^+ \quad \text{(by assumption)} \\
&= [K e^{rj} - S_{t+j} e^{\delta j}]^+ \quad \text{(given the drift to the asset)} \\
&\geq [K - S_t]^+ \quad \text{(if } r > \delta \text{ as assumed)}
\end{aligned}
$$

Since $G_{t,t+j} \geq [K-S_{t+j}]^+ = X_{t+j}$, for $0 \leq j < T$, then by Lemma 2, $G_{t,t+j} \geq V_{t+j}$. *QED*.

Our specifications so far for the bounding claim G have been of European type. One key reason why European type bounding claims may be preferred is because European claims are valued less than their American counterparts and as such are likely to provide tighter upper bounds for the standard American option value. Further, if G is of American type but its (intrinsic) value may exceed its expected maturity payoff in violation of Lemma 3, then it does not help the cause of skipping the computation of an American claim. Consider, for example, the ASPE bound. While its expected maturity payoff or pure option value, $G_{CY,t} = E_t(K - e^{-(r-\delta)(T-t)}S_T)^+$, never falls below $[K-S_t]^+$, there is no guarantee that $G_{CY,t}$ will not below its own intrinsic value $(K - e^{-(r-\delta)(T-t)}S_t)^+$, if it were of American type. This is because it's intrinsic value $(K - e^{-(r-\delta)(T-t)}S_t)^+$ also exceeds $[K-S_t]^+$ as long as the asset has a positive risk-neutral drift ($r>\delta$).

Consider a numerical example to see this. Suppose the Black–Scholes setup applies with current time $t = 0$, maturity time $T = 3$, current stock price $S_t = \$80$, strike price $K = \$100$, constant risk-free rate $r = 10\%$, dividend yield or leakage rate $\delta = 0\%$, and the constant volatility rate $\sigma = 30\%$. Using these values, the maturity payoff function of Chen and Yeh's G is $X_{G,T} = \text{Max}(0,100-0.7408\ S_T)$. The (risk-neutral) expected maturity payoff of G at time t, namely the pure European option value of G is $E_t[X_{G,T}] = \$30.08$. It is above the \$20 intrinsic value of the standard American option and indeed it is an upper bound for the standard American option value of $V_t = \$21.27$. [17] However, $E_t[X_{G,T}]$ is below its own intrinsic value, $X_{G,t} = \text{Max}(0,100-0.7408*80) = 40.73$.

Therefore, if the G is of American type, then it is potentially useful as an upper bound for V only if G satisfies Lemma 3, since in that case the upper bound, namely the expected maturity payoff of G, should be easy to calculate.

We now propose generalized American claims, the expected maturity payoff of which can be used as upper bounds. In this context, we abstract from randomizing the terminal payoff. As mentioned earlier, the early exercise value and premium of an American option becomes difficult to interpret, if at all possible, when dealing with a randomized payoff function. Further, for simplicity, the generalization involves only the number of optioned asset units. The following lemma presents a class of bounding American claims under these circumstances.

Lemma 5. Suppose, at time t, the intrinsic value of the generalized American put option G is $X_{G,t} = \text{Max}(0,\ K-\lambda_t S_t)$, where $0 < \lambda_t \leq 1$ is a

deterministic monotonically decreasing function of time (i.e., $\partial\lambda_t/\partial t<0$) and $\lambda_T<\lambda_t S_t/E_t(S_T)$. Then, the expected maturity payoff, $E_t(X_{G,T})$, is an upper bound for the conventional American spot put option value, V_t.

Proof: Let t^* be the optimal stopping time for the conventional spot put option V. Then,

$$\begin{aligned} V_t &= \boldsymbol{E}_t\left[\left(\exp\left(-\int_t^{t*} r_u \mathrm{d}u\right)\right)(K-S_{t*})^+\right] \\ &= \boldsymbol{E}_t\left[\left(\exp\left(-\int_t^{t*} r_u \mathrm{d}u\right)\right)\{(K-\lambda_{t*}S_{t*})+S_{t*}(\lambda_{t*}-1)\}^+\right] \\ &\le \boldsymbol{E}_t\left[\left(\exp\left(-\int_t^{t*} r_u \mathrm{d}u\right)\right)(K-\lambda_{t*}S_{t*})^+\right] \\ &\le G_t^A \end{aligned}$$

In the first inequality, we have used the property that for real numbers a and b, $(a^+ + b^+) \ge (a+b)^+$, and that $0<\lambda_{t*}\le 1$. The second inequality follows from the fact that t^* is the optimal stopping time for V and not for G. So far, we have shown that the generalized claim's American option value is an upper bound for the conventional American put option.

Now, the generalized claim G's pure European option value or expected maturity payoff is $\boldsymbol{E}_t(K-\lambda_T S_T)^+$. By the convexity of payoff, we have:

$$\begin{aligned} \boldsymbol{E}_t[K-\lambda_T S_T]^+ &\ge [K-\lambda_T \boldsymbol{E}_t(S_T)]^+ \\ = [K-\lambda_t S_t\{\lambda_T \boldsymbol{E}_t(S_T)/\lambda_t S_t\}]^+ &\ge [K-\lambda_t S_t]^+ \end{aligned}$$

The last inequality follows from the restriction $0<\lambda_T<\lambda_t S_t/\boldsymbol{E}_t(\boldsymbol{S_T})$ or $0<\lambda_T \boldsymbol{E}_t(\boldsymbol{S_T})/\lambda_t S_t<1$. Thus the pure European option value or expected maturity payoff of G never dips below its intrinsic value. This means the pure American value of G equals its expected maturity payoff. Since the pure American value of G is greater than the American value of G (G_t^A), its expected maturity payoff is an upper bound for G_t^A. As we have shown that G_t^A is an upper bound for V_t, it follows that the generalized claim G's pure European option value or expected maturity payoff is indeed an upper bound for the American spot option value V_t. *QED*.

Note that in the above proof, we did not use any specific assumption about the stochastic processes of the underlying asset price and its volatility. Nor did we make any assumption about the stochastic process for the risk-free rate or the leakage rate. Thus, Lemma 5 applies to arbitrary risk-neutral stochastic processes.

4. IMPLICATIONS OF UPPER BOUNDS

Among the various implications of the upper bound requirements and specifications that we have discussed so far, we focus here on two areas. *First*, we discuss the implications for relative pricing of options. *Second*, relevance in the context of empirical option pricing is explored.

4.1. Relative Pricing of Options

Options of different strikes and maturities on an asset are traded as separate securities but they share the same underlying stochastic process. This forces many restrictions on rational (arbitrage-free) pricing of options relative to the underlying asset and relative to each other. Most primitive of these are the upper and lower bounds on individual option prices reflecting option pricing relative to the asset. Put-Call Parity (or bounds) imposes pricing discipline on call and put options of the same strike relative to the asset. Pricing conditions concerning options alone are numerous. Some examples are lower vs. higher strike, shorter vs. longer maturity, combinations of call and put options, etc. Ideally, upper bounds for options should retain such relative pricing discipline.

While a full investigation is beyond the scope of this paper, we examine below two arbitrage conditions that relate to put options of different strikes to get a sense of whether the relative pricing conditions of option prices carry over to their upper bounds. For this purpose, we assume a constant interest rate and set the leakage rate to zero, and we use the nonrandomized version of the generalized European option as an upper bound with the number of optioned assets set to one: $G_t = e^{-r(T-t)}E_t[e^{r(T-t)}K - S_T]^+$.

4.1.1. Lower vs. Higher Strike American Put Options

Suppose we have two American put options V_1 and V_2 with strike prices $K_1 > K_2$, both maturing at time T. One arbitrage condition between the prices of these two options is that $V_1 > V_2$. The following corollary shows that this basic pricing discipline is carried over to their generalized European upper bounds.

Corollary 3. Let G_{1t} and G_{2t} be the generalized European option upper bounds for the T-maturity standard American put option prices V_{1t} and V_{2t} with strike prices $K_1 > K_2$: $G_{1t} = e^{-r(T-t)}E_t[e^{r(T-t)}K_1 - S_T]^+$, $G_{2t} = e^{-r(T-t)}E_t[e^{r(T-t)}K_2 - S_T]^+$. Then, $G_{1t} \geq G_{2t}$.

Proof: It suffices to show that $\boldsymbol{E}_t[e^{r(T-t)}K_2-S_T]^+ \leq \boldsymbol{E}_t[e^{r(T-t)}K_1-S_T]^+$

$$\begin{aligned}\boldsymbol{E}_t[e^{r(T-t)}K_2 - S_T]^+ &= \boldsymbol{E}_t[e^{r(T-t)}\{K_1 + (K_2 - K_1)\} - S_T]^+ \\ &= \boldsymbol{E}_t[\{e^{r(T-t)}K_1 - S_T\} + e^{r(T-t)}(K_2 - K_1)]^+ \\ &\leq \boldsymbol{E}_t[e^{r(T-t)}K_1 - S_T]^+ + e^{r(T-t)}(K_2 - K_1)^+ \\ &= \boldsymbol{E}_t[e^{r(T-t)}K_1 - S_T]^+.QED.\end{aligned}$$

In the first inequality, we have used the property that for real numbers a and b, $(a+b)^+ \leq (a^+ + b^+)$. The last equality follows from $K_1 > K_2$. Further, G_{1t} and G_{2t} are themselves tradable European options. Therefore, in an arbitrage-free market, $G_{1t} \leq G_{2t}$ cannot prevail. To see this, suppose $G_{1t} \leq G_{2t}$. Then, sell one G_2 option and buy one G_1 option. The net proceed now is $(G_{2t} - G_{1t}) \geq 0$. At maturity time T: if $S_T < K_2$, the payoff is $+(K_1 - K_2)$, if $K_2 < S_T \leq K_1$, the payoff is $+(K_1 - S_T)$, and if $K_1 < S_T$, the payoff is 0.

Thus the arbitrage strategy leads to nonnegative proceeds now, nonnegative payoff at maturity, and nonzero probability of positive payoff at maturity. Therefore, in the absence of arbitrage, $G_{1t} \geq G_{2t}$ should prevail.

4.1.2. Put Option (Money) Spreads

Suppose we have two American put options V_1 and V_2 with strike prices $K_1 > K_2$, both maturing at time T. An important arbitrage condition on the prices of these two options is that, in the absence of arbitrage, the long bear spread cannot be worth more than the difference in strikes, i.e., $(V_1 - V_2) < (K_1 - K_2)$. The following corollary shows that this pricing discipline is carried over to their generalized European upper bounds.

Corollary 4. Let G_{1t} and G_{2t} be the generalized European option upper bounds for the T-maturity standard American put option prices V_{1t} and V_{2t} with strike prices $K_1 > K_2$: $G_{1t} = e^{-r(T-t)}E_t[e^{r(T-t)}K_1 - S_T]^+$, $G_{2t} = e^{-r(T-t)}E_t[e^{r(T-t)}K_2 - S_T]^+$. Then, $(G_{1t} - G_{2t}) \leq (K_1 - K_2)$.

Proof:

$$\begin{aligned}G_{1t} &= e^{-r(T-t)}E_t[e^{r(T-t)}K_1 - S_T]^+ = e^{-r(T-t)}E_t[e^{r(T-t)}(K_1 - K_2) \\ &\quad + (e^{r(T-t)}K_2 - S_T)]^+ \\ &\quad \leq (K_1 - K_2) + e^{-r(T-t)}E_t[e^{r(T-t)}K_2 - S_T]^+ \\ &= (K_1 - K_2) + G_{2t} \\ &\quad \Rightarrow (G_{1t} - G_{2t}) \leq (K_1 - K_2).QED\end{aligned}$$

It is perhaps premature to say that all arbitrage conditions involving the American options would carryover to the generalized European upper bounds. However, in light of the fact that these upper bounds here are European options and based on Corollaries 3 and 4 above, we are optimistic that the rational option pricing bounds would apply to the upper bounds. The importance of this carryover property for empirical option pricing will be discussed shortly.

4.1.3. Trading Early Exercise Options

A standard American option is a package of a standard European option and an early exercise option. In practice, we observe trading of American options but not the early exercise options separately. Based on Margrabe (1978) and our analysis in this paper, it looks like one can trade the early exercise options indirectly using the European options alone. To see this, we first present an implication of the generalized European claims in this regard.

Corollary 5. Let G_t be the generalized European option upper bound for the T-maturity standard American put option value V_t with strike prices K:

$$G_t = e^{-r(T-t)}E_t[e^{r(T-t)}K - S_T]^+$$

Then, (a) there exists a standard European option of strike K^*, with $K < K^* < e^{r(T-t)}K$, such that its value v^*_t equals the American option value V_t, and (b) the early exercise feature of the American option is valued at $EEP_K = v^*_t - v_t$, where $v_t = e^{-r(T-t)}E_t[K-S_T]^+$ is the value of a standard European option of strike K.

Part (*a*) of the statement above follows directly from the fact that the European option value is a monotonic increasing function of the strike price, while part (*b*) simply reflects the two components of an American option value.

A long put spread strategy involves a long position in the higher strike put option and a short position in the lower strike put option. A striking interpretation of Corollary 5 is that all long put (money) European spreads essentially represent ratio positions in the early exercise option associated with the lower strike. While determination of the strike K^* is equivalent to calculating the American option value V_t and thus provides no computational relief, the economic insight is that it is not necessary to trade American options in order to trade the early exercise option. One practical difficulty in using the spread in lieu of the American option itself is that the investor needs to dynamically adjust the strike K^*, or equivalently the ratio in the spread.

4.2. Relevance for Empirical Option Pricing

A clear strength of the arbitrage-based theoretical models of option pricing is that by definition they incorporate arbitrage-free relative pricing of the asset and all derivatives including the options. These relative pricing restrictions are obviously of greater importance in the context of American options because of their early exercise feature. However, the recent parametric theoretical models are already quite complex to implement in the context of European options. As such, in empirical studies of American options using parametric models, the relative pricing bounds do not receive much attention either.

The European claims that we have proposed as upper bounds for American options should be helpful in empirical testing of parametric American option pricing models. For example, one can estimate the parameters of the asset price process from the observed asset returns, form the risk neutral terminal distribution given the theoretical valuation model, and then compute the value of the bounding generalized European claims using the risk-neutral distribution. To the extent the risk-neutral return dynamics is properly captured by the theoretical model, the estimated upper bounds should all be above the observed American option prices. If this leads to a failure of the theoretical model, then the much more complex task of estimating the theoretical American option prices may not be worthwhile.

Given the limited nature of success of the various parametric theoretical extensions of the Black–Scholes model in explaining the patterns of observed option prices, a number of researchers have explored nonparametric alternatives. These nonparametric methods attempt to extract an empirical option valuation model from the actual option prices themselves. Semiparametric versions arise when guidance from some theoretical model(s) is used to improve dimensionality of the estimation problem. For example, Ait-Sahalia and Lo (1998) estimate empirical pricing function for European options on the S&P 500 Index and the implied state price density using nonparametric and semiparametric methods. Broadie, Detemple, Ghysels, and Torres (2000), on the other hand, study the properties of nonparametric empirical American option pricing function and exercise boundary for the S&P 100 Index options.

It seems that nonparametric studies such as the above do not quite consider whether the estimated pricing functions obey the various arbitrage bounds including the upper and lower bounds for option prices.[18] Imposing or testing for these bounds is even more important for nonparametric models, especially in the context of American options, as there is no built-in

arbitrage-free structure of prices here. It is hoped that the upper bounds can be helpful in controlling the quality of nonparametric option models.

For example, suppose the researcher estimates an empirical European option pricing function for the S&P 500 (SPX) and an empirical American option pricing function for the S&P 100 (OEX) using kernels on several predictors including moneyness and volatility. Based on our results, adjusting for slight changes in volatility and leakage and for the index level, the option price predicted for a K-Strike T-Maturity OEX put option should be lower than the predicted price for a T-maturity SPX put option with strike $K^u = Ke^{r(T-t)}$. If not, the empirical pricing functions are such that would permit arbitrage across the SPX and OEX contracts.

The CBOE has of late introduced European option contracts (XEO) on the S&P 100 Index. As the volume and open interest of the XEO options grow, the arbitrage (upper) bounds tests will likely become easier in future. Later in this paper we examine sample quotes for these options.

5. A QUASI-BOUND

A potential weakness of upper bounds that do not rely on approximating the early exercise boundary is that the bounds may be quite wide. In the context of the generalized European claims in a Black–Scholes setup, we now propose a claim that holds as an upper bound except for a range of moneyness not commonly traded on organized exchanges.

Corollary 6. Suppose Q is a European put option with the following maturity payoff function: $X_{Q,T} = [K - \lambda_T S_T]^+$, where $\lambda_T = [S_t - K\{1 - E_t R_{t,T-t}\}]/E_t[R_{t,T-t} S_T]$, and $R_{t,T-t} = \exp(\int^T_t ru\mathrm{d}u)$ is the discount factor between t and T. Then, the European put option Q's value $Q_t = E_t[R_{t,T-t}(K - \lambda_T S_T)^+]$ is an upper bound for the standard American put option's value V_t and Q is meaningfully defined as a put option for $(S_t/K) > (1 - E_t R_{t,T-t})$.

Proof:

$$Q_t = E_t[R_{t,T-t}(K - \lambda_T S_T)]^+ \geq [KE_t R_{t,T-t} - \lambda_T E_t\{R_{t,T-t} ST\}]^+$$
$$= [K - S_t]^+ (\text{using the given value of } \lambda_T)$$

By Lemma 2, then $Q_t \geq V_t$ and Q is meaningfully defined as a put option as long as $\lambda_T > 0$, that is $(S_t/K) > (1 - E_t R_{t,T-t})$. *QED*.

The European claim Q is a quasi-bound since it is not meaningfully defined as a put option when the American put option is too deep-in-the-money, that

is the compound interest value on K is too high. For longer maturity options, Q reaches this threshold level sooner than for shorter maturity options. However, this shortcoming is not practically that important since below the threshold, Q_t can be set to the American option's intrinsic value.

The reason Q tightens the AKE and ASPE bounds is because Q adjusts the number of optioned units (λ_T) of the underlying asset depending on the moneyness of the option. While the AKE and ASPE adjustments are fixed for a time to maturity, λ_T decreases (increases) with the moneyness of the put option (asset price). For at-the-money put options ($S_t = K$), the adjustment factor λ_T of Q is equal to the adjustment factor of ASPE, and for in-the-money (out-of-the-money) put options λ_T is lower (higher).

To have a general feeling about the bounds, let us now present some numerical results for the Black–Scholes setup: constant interest rate of 10%, zero leakage rate, time to maturity of 0.25 years, and constant volatility of 30%. The strike price is set to 100 and the asset price is varied from 80 to 120. The American put option price is calculated using a 100-step Binomial Tree. Fig. 1 plots five series against the measure of moneyness

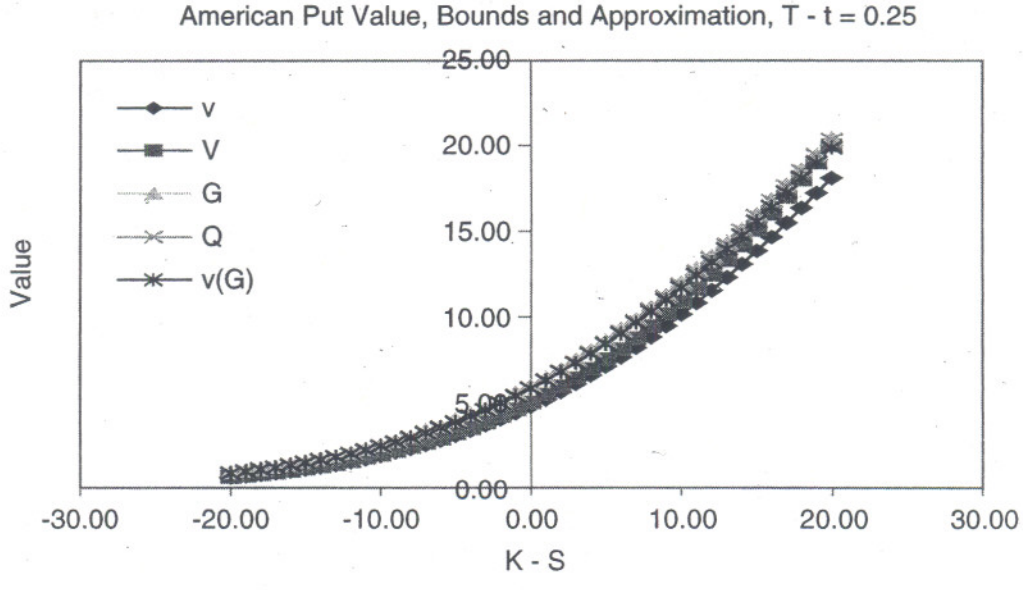

Fig. 1. Black–Scholes Setup: Put Option Values, Bounds, and Approximation. *Note:* The parameter values used for this chart are: interest rate $r = 10\%$, leakage rate $\delta = 0\%$, volatility $\sigma = 30\%$, time to maturity $T-t = 0.25$ *year*, and $K = 100$. The legends are as follows: v = European value, V = American value, G = Generalized European option value, Q = Quasi-Bound of this paper, and $v(G)$ = discounted value of G. The American option value is calculated using a 100-step Binomial Tree.

$K-S$: v (European option value), V (American option value), G (the ASPE bound), Q (the Quasi-Bound value), and $v(G)$ (discounted value of the ASPE bound). Although $v(G)$ is not an upper bound, we have included $v(G)$ to see how well this approximates the American option value V.

Several observations can be made from Fig. 1. First, as expected, G indeed bounds V from above and so does Q given the parameter values. Second, the curvature of all the bounds and the approximation (G, Q, and $v(G)$) are very similar to that of the American option. This is rather encouraging as the hedge ratios based on the bounds and the approximations are expected to be good estimates for the true hedge ratio. Third, the bounds and the approximation track the American option value very closely for in-the-money put options. This is also encouraging since in practice in-the-money observed option prices are believed to be notoriously unreliable. The bounds and the approximation here can thus provide good valuation guidance for these options. Fourth, as expected, the quasi-bound (Q) provides a tighter bound than the ASPE bound. Fifth, the discounted value $v(G)$ of the ASPE bound provides a nice approximation although its theoretical relationship to V is unclear.

The results in Fig. 1 are for short maturity options. Fig. 2 presents the results for time to maturity of 1.0 year, other parameters remaining the same

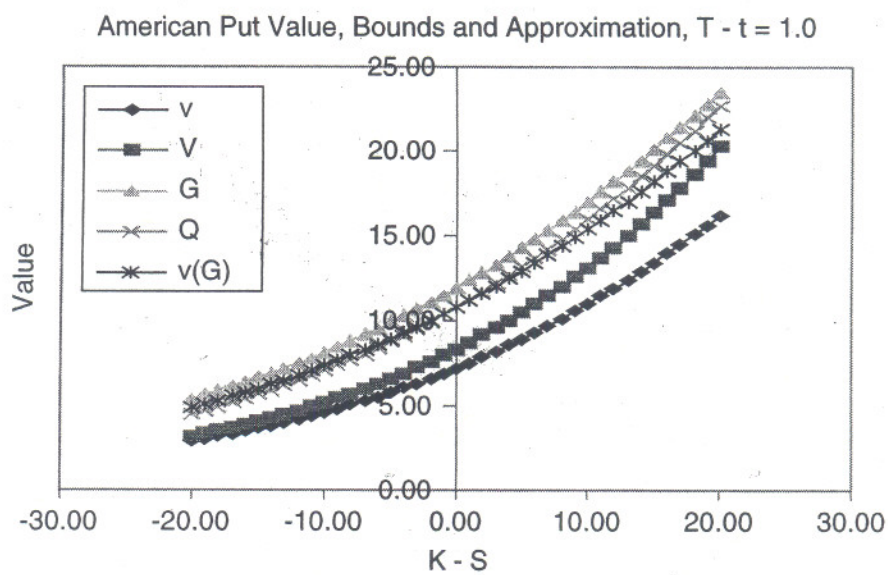

Fig. 2. Black–Scholes Setup: Put Option Values, Bounds, and Approximation. *Note:* The parameter values used for this chart are: interest rate $r = 10\%$, leakage rate $\delta = 0\%$, volatility $\sigma = 30\%$, time to maturity $T-t = 1.0$ *year*, and $K = 100$. The legends are as follows: v = European value, V = American value, G = Generalized European option value, Q = Quasi-Bound of this paper, and $v(G)$ = discounted value of G. The American option value is calculated using a 100-step Binomial Tree.

as in Fig. 1. As expected the bounds and the approximation widen relative to the American option value with a substantially longer maturity as they do not consider the true exercise boundary and overestimate the expected interest value. However, both G and Q still track the curvature well. As the option goes deep in-the-money, the American option's hedge ratio approaches –1.0 faster than the bounds and the approximation. Once again this is due to the fact that the intrinsic value of the bounds here always stays above the intrinsic value of the American option by design. It is also to be noted that for deep-in-the-money option, the Quasi-Bound Q hits its threshold level with the longer time to maturity and the $v(G)$ approximation deteriorates as well.

Next we consider the joint effect of a lower volatility (15%) and a lower interest rate (5%) in Figs. 3 and 4. Unlike the European option component, the early exercise component of the American option tends to go up with a lower volatility. Lowering the interest rate of course reduces the value of the American put option. The 50% reduction in both the volatility and the interest rate, however, reduced the American option value in the current experiment. As expected, the bounds and the approximation seem to track

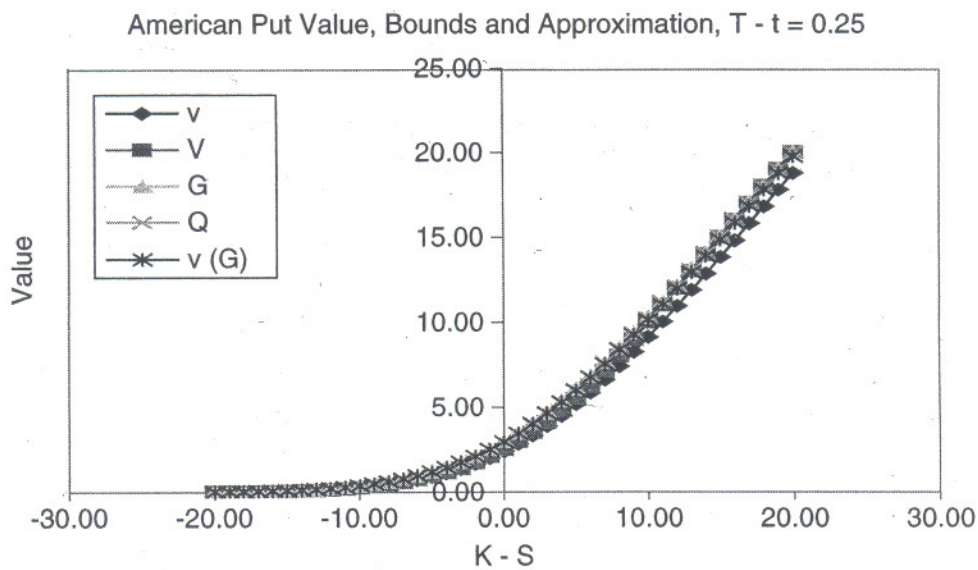

Fig. 3. Black–Scholes Setup: Put Option Values, Bounds, and Approximation. *Note:* The parameter values used for this chart are: interest rate $r = 5\%$, leakage rate $\delta = 0\%$, volatility $\sigma = 15\%$, time to maturity $T-t = 0.25$ *year*, and $K = 100$. The legends are as follows: v = European value, V = American value, G = Generalized European option value, Q = Quasi-Bound of this paper, and $v(G)$ = discounted value of G. The American option value is calculated using a 100-step Binomial Tree.

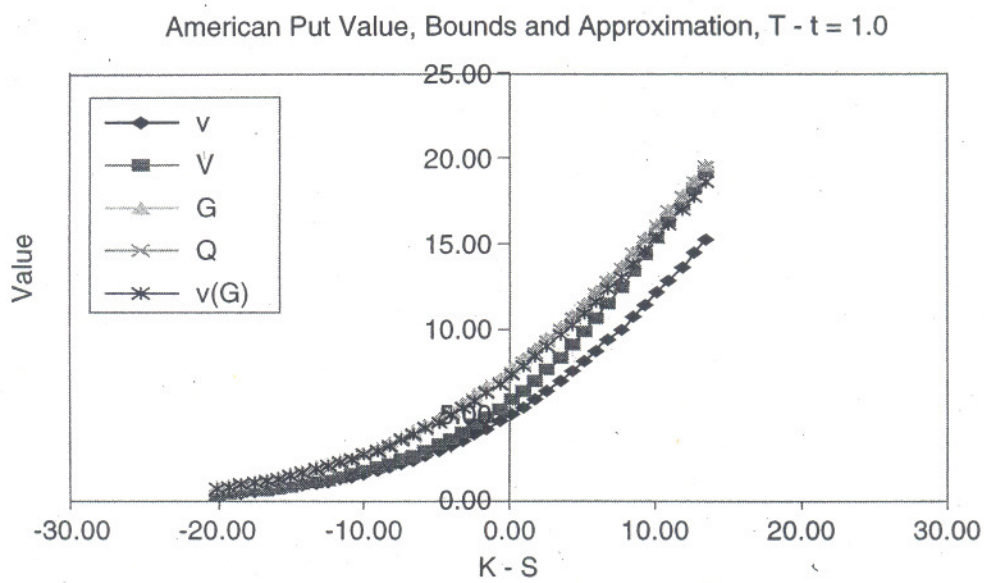

Fig. 4. Black–Scholes Setup: Put Option Values, Bounds, and Approximation. *Note:* The parameter values used for this chart are: interest rate $r = 5\%$, leakage rate $\delta = 0\%$, volatility $\sigma = 15\%$, time to maturity $T-t = 1.0$ *year*, and $K = 100$. The legends are as follows: v = European value, V = American value, G = Generalized European option value, Q = Quasi-Bound of this paper, and $v(G)$ = discounted value of G. The American option value is calculated using a 100-step Binomial Tree.

the American option value better with a lower volatility–lower interest rate combination, especially for the short maturity options.

Overall, it appears that the European nature of the bounds and the approximation help retain the essential convexity-of-payoff related properties of American option values. However, one weakness that needs further attention is that the American option value is more convex than the bounds and the approximation and it approaches the intrinsic value faster as the option goes deeper in-the-money.

6. SAMPLE S&P 100 OPTION QUOTES

The purpose of this section is to see, on a very preliminary basis, if the upper bound properties hold in practice. To our knowledge, only the S&P 100 Index has both American (OEX) and European (XEO) option contracts available. This should greatly facilitate empirical study of American options, their bounds, and the nature of early exercise premium (EEP). However, the

European (XEO) contracts are relatively new and their volume is currently far less than that of the well-known American (OEX) contracts.[19] Meantime, the bid–ask quotes of the XEO and OEX contracts can still provide useful insights into the pricing of American options vis-à-vis their European counterparts. In particular, it will be interesting to see if the European option based bounds proposed in this paper apply to the American options in practice.

In Panel A of Table 1, we report a sample of CBOE option quotes for the XEO and OEX June, 2002 contracts. The Bid and Ask quotes are 15-minute delayed quotes retrieved from the CBOE web site at about 1:56 PM on March 14, 2002; the S&P 100 Index was hovering about the 585.00 level around that time (largely unchanged from its level around 1:41 PM).

Given that the XEO market is not as liquid as the OEX market, the last sale prices of the OEX and XEO contracts may not match. Also, the last sale prices of the XEO and OEX contracts of various strikes may not be comparable. But the Bid and Ask quotes are updated much more frequently and as such are more representative of the respective option values. In line with empirical option pricing tradition, we take the mid-point of the Bid–Ask spread as an estimate of the fair price of the option. To see if the observed American option price is bounded by the price of a compounded strike European option, we estimate the compounded strike, $K^u = Ke^{r(T-t)}$, using a risk-free rate of 5% and the 0.3671 year time to maturity of the options. Luckily, the compounded strike is fairly close to the next available strike of the sample options and as such the observed quote of the XEO option closest to the compounded strike K^u can be used as a proxy for the upper bound of the K-strike OEX option.

It seems that the observed American option (mid) quotes are indeed bounded by the corresponding compounded strike European option counterparts. For example, the $K = 570$ OEX contract's mid-quote \$12.75 is less than the $K = 580$ ($K^u = 581$) XEO contract's mid-quote \$15.95. Similarly, the $K = 580$ OEX contract's mid-quote \$16.35 is less than the $K = 590$ ($K^u = 591$) XEO contract's mid-quote \$20.35.

Panel A data also provides an opportunity to see if the observed American option spread is bounded by the compounded strike European option spread. The long spread reported against a strike, say 570 ($K^u = 581$), represents the net cost of buying the option of that strike (570, $K^u = 581$) at the Ask quote and selling the immediately lower strike (560, $K^u = 570$) option at the Bid quote. The long XEO spread (long 580, short 570) cost reported against 580 is then used as a proxy for bounding the long OEX spread (long 570, short 560) cost reported against 570. Indeed, the observed bounding

Table 1. March 14, 2002 CBOE Option Quotes for the S&P 100 Options.

Panel A

CBOE 15-Minutes Delayed June-Maturity Put Option Quotes for OEX and XEO

Trading Date: March 14, 2002 — r = 0.0500

Quotes Retreived from CBOE Site at about 1:56 PM — $T-t$ = 0.3671

S&P 100 Index Around 585.00 During 1:40 and 2:00 PM — $\exp[r(T-t)]$ = 1.0185

K	XEO Bid	XEO Ask	XEO Mid	XEO Ask–Bid	OEX Bid	OEX Ask	OEX Mid	OEX Ask–Bid	K^* $\exp[r(T-t)]$	Long XEO Spread	Long OEX Spread
550	7.00	7.70	7.35	0.70	7.30	8.00	7.65	0.70	560		
560	9.20	9.90	9.55	0.70	9.50	10.20	9.85	0.70	570		2.90
570	11.60	13.10	12.35	1.50	12.00	13.50	12.75	1.50	581	3.90	4.00
580	15.20	16.70	15.95	1.50	15.60	17.10	16.35	1.50	591	5.10	5.10
590	19.60	21.10	20.35	1.50	20.00	22.20	21.10	2.20	601	5.90	6.60
600	24.60	26.80	25.70	2.20	25.30	27.50	26.40	2.20	611	7.20	7.50
610	31.00	33.20	32.10	2.20	31.70	33.90	32.80	2.20	621	8.60	
620	38.20	40.40	39.30	2.20	39.10	41.30	40.20	2.20			

Panel B

K	EEP Mid = OEX–Mid Less XEO–Mid	EEP Ask = OEX–Ask Less XEO–Bid	EEP Bid = OEX–Bid Less XEO–Ask	EEP Ask–Bid
550	0.30	1.00	−0.40	1.40
560	0.30	1.00	−0.40	1.40
570	0.40	1.90	−1.10	3.00
580	0.40	1.90	−1.10	3.00
590	0.75	2.60	−1.10	3.70
600	0.70	2.90	−1.50	4.40
610	0.70	2.90	−1.50	4.40
620	0.90	3.10	−1.30	4.40

Panel A of this table contains the Bid and Ask quotes for the S&P 100 European options (XEO) and American options (OEX) maturing in June, 2002. These 15-min delayed quotes were retrieved from the CBOE web site at about 1:56 PM; the S&P 100 Index was about 585.00 around that time (largely unchanged from its level around 1:41 PM). The mid-point of the Bid–Ask spread is an estimate of the fair value of the option. The compounded strike $K\exp[r(T-t)]$ is estimated using a risk-free rate of 5% and the 0.3671 year time to maturity of the June options (it seems to be fairly close to the next available strike). The long spread reported against a strike (say 570) represents the net cost of buying the option of that strike (570) at the Ask quote and selling the immediately lower strike (560) option at the Bid quote.
Panel B of this table first estimates the early exercise premium (EEP) as the difference between the mid quotes of the OEX and XEO options. The EEP Ask quote is then estimated as the net cost of buying the OEX option at the Ask quote and selling the XEO option of same strike at the Bid quote. The EEP Bid quote is estimated as the net proceeds of selling the OEX option at the Bid quote and buying the XEO option of same strike at the Ask quote.

XEO spread cost of \$5.10 is greater than the OEX spread cost of \$4.00. This is also the case for spreads involving other strikes.

In Panel B of Table 1, we estimate the EEP of a given strike as the difference between the mid-quotes of the OEX and XEO contracts. The magnitude of the EEP is not large. However, the behavior of the estimated EEP is largely in line with theory. The estimated EEP seems to increase with the strike price and the increment appears larger for deeper in-the-money options.

Given the existence of both European (XEO) and American (OEX) options, one can trade the EE option synthetically. The cost of buying a synthetic EE option (EEP Ask) is estimated as the OEX Ask net of the XEO Bid. Similarly, the proceeds of selling a synthetic EE option (EEP Bid) are estimated as the OEX Bid net of the XEO Ask. From the sample information in Panel B of Table 1, buying a synthetic EE option seems quite expensive while shorting a synthetic EE option is not feasible at all. This may in part explain why the XEO contracts are not as popular as the OEX contracts although the European options are cheaper and should have attracted more speculators, hedgers, and portfolio insurers.

The primary reason for the synthetic EE option anomaly is that the (dollar) Bid–Ask spreads for the index option (both XEO and OEX) contracts are too wide relative to the size of the EEP. For example, in Panel B of Table 1, the EEP is estimated at about \$0.40 for $K = 570$ but the Bid–Ask spread is \$1.50 for both XEO and OEX contracts. This poses a challenging measurement problem for empirical options researchers. Further, a policy question also arises as to whether the CBOE should act to sufficiently reduce the spreads in both types of contracts so that investors are not limited to only long positions in (synthetic) EE options. One way to make this possible is to replace the OEX contracts with EE contracts. For example, an EE contract can be designed to pay the buyer the excess of the intrinsic value over the XEO mid-quote in case the EE option is exercised. This suggestion is in line with the fact that in the presence of transaction costs, synthetic replication may not closely track the value of directly tradable derivatives. Further, the European component of the OEX contract is clearly redundant given the XEO contract.

7. SUMMARY AND CONCLUSIONS

This paper has provided a fuller characterization of the analytical upper bounds for American options by establishing properties that are required of the bounds. A key property is that if a claim's value never falls below the intrinsic value of the American option, then the claim is an upper bound for

the American option value. While the literature primarily relies upon bounds for the exercise boundary, we have shown that a class of generalized European options can be made to satisfy the key property and hence serve as upper bounds. This class contains the analytical bounds of Margrabe (1978) and Chen and Yeh (2002).

An important benefit of having generalized European options as upper bounds is that they are in closed form and are easy to implement since a direct treatment of the early exercise boundary is avoided. They are also intuitively tractable. When the valuation situation involves multiple state variables, the class of European upper bounds suggested in this paper can significantly ease the burden of computation and still serve as useful benchmarks.[20] This characteristic is also quite beneficial in practice where only a general valuation range is desired.

The upper bounds seem to have many desirable properties and interesting implications. For example, we have shown that the across-strike arbitrage conditions on option prices seem to carry over into the bounds and that one can trade early exercise options using merely European option spreads and never trading the American options. We believe both parametric and nonparametric empirical option pricing models can improve the quality of estimation using the bounds results of this paper. So far empirical attempts to incorporate various arbitrage bounds have been lacking, especially in nonparametric models.

In an attempt to tighten the European-type bound, we proposed a quasi-bound that holds as an upper bound for most practical circumstances. We also suggest an approximation based on the bound of Chen and Yeh (2002). Our limited numerical results in the traditional Black–Scholes setup are encouraging. The bounds and the approximation of this paper seem to track the American option value and its curvature rather well for short maturity options and in-the-money options. Hence the bounds here should be useful in estimating American option hedge ratios and as valuation benchmarks or proxies for in-the-money options for which observed option prices are believed to be notoriously unreliable. A caveat is that as the maturity gets longer and the volatility and interest rate increase, the bounds and the approximation widen relative to the American option value. Another potential weakness that needs further attention is that the European-type bounds and approximation do not change fast enough as the American option goes deep in-the-money. This is, of course, a tradeoff that arises from not explicitly considering the early exercise boundary of the American option.

In this paper we did not undertake any empirical study of the upper bounds. However, as a first attempt, we did examine sample (March 14, 2002) option quotes for the European (XEO) and the American (OEX)

options on the S&P 100 Index. These quotes appear well behaved with respect to the upper bound properties. But a synthetic long position in the early exercise option seems quite expensive and a synthetic short position in the early exercise option is not feasible. This is because the bid–ask spreads are too wide relative to the magnitude of the EEP, and is likely one of the factors why the XEO contracts are not as popular as the OEX contracts. Redesigning the OEX contracts purely as early exercise options would eliminate the redundancy of the European option (XEO) and allow investors to take long as well as short positions in the early exercise option directly.

NOTES

1. Analytic solutions that require numerical evaluation of the early exercise boundary are available in Kim (1990), Jamshidian (1992), Geske and Johnson (1984), Jacka (1991), Carr, Jarrow, and Myneni (1992), Bunch and Johnson (2000), and Broadie, Detemple, Ghysels, and Torres (2000). Analytic approximations based on approximation of the early exercise boundary include Johnson (1983), Omberg (1987), Huang, Subrahmanyam, and Yu (1996), Ju (1998), Broadie and Detemple (1996), and Bunch and Johnson (2000). Broadie and Detemple (2004) provide a recent survey of American option valuation.

2. For example, in nonparametric estimation of American option pricing function, upper bounds should be useful in controlling the quality of estimation.

3. Based on the upper bound of Chaudhury and Wei (1994), Chaudhury (1995) developed several Black-Scholes type closed form analytic approximations for American futures options that are quite accurate and provide better approximation than the quadratic approximation of MacMillan (1986) and Barone-Adesi and Whaley (1987) for most actively traded futures options.

4. Applications of analytical upper bound for American futures option prices (Chaudhury & Wei, 1994) include Bates (2000), Melick and Thomas (1997), Leahy and Thomas (1996), Soderlind and Svensson (1997), Flamouris and Giamouridis, (2002), Galeti and Melick (2002), and Beber and Brandt (2003). The bounds of this paper will be helpful in extracting information from American spot options data.

5. References on option bounds based on equilibrium pricing kernel can found in Huang (2004).

6. Closed form bounds require closed form terminal distribution of the asset price.

7. For example, in a two-factor random volatility model, the upper bound of Chen and Yeh (2002) is more than 3700 times faster than the American finite difference algorithm.

8. For example, Chen and Yeh (2002) have given several examples involving stochastic interest rates, leakage, and volatility. The bounds of this paper also apply to such cases. Both in Chen and Yeh and in this paper, the only requirements are that: (a) the risk neutral measure exists, (b) the values of the stochastic discount factor are less than one for all sample paths, and (c) the instantaneous expected net growth process is strictly positive (to make the American spot put option problem interesting).

9. This is the argument used by Chaudhury and Wei (1994) and Chaudhury (1995) for American futures options. For these options, the pure European value always stays above the intrinsic value (Lieu, 1990; Chen & Scott, 1992).

10. Unless mentioned otherwise, all expectations in this paper are expectations under the risk-neutral or equivalent martingale measure.

11. Chen and Yeh (2002, p. 119 and FootNote 4, p. 120) recognize these limitations of their Theorem 1.

12. Chen and Yeh (2002, p. 118) mention that an upper bound for an American option always stays above both the continuation and the exercise value of the American option. Of course, this is definitional of an upper bound.

13. See Chen and Yeh (2001) for the treatment of stochastic interest rates. For interested readers, the author of this paper can provide the proof that the results here are unaffected by stochastic interest rates and leakage.

14. Examples of further drift adjustment include Bakshi, Cao and Chen (1997) and Bates (2000) for jumps in asset price in a stochastic volatility framework.

15. For $r > \delta$, maturity payoff of Chung and Chang's (2005) bound is equivalent to equal adjustments to the strike price and the number of units of the optioned asset; in that case, their bound translates to adjusting the number of standard or conventional European options on the asset. For $r < \delta$, their bound is like European option on $\exp(\delta T)$ units of stocks for a total strike of $K \exp(rT)$, that is an implied strike of $K \exp\{(r-\delta)T\} < K$ per unit of stock; in this case, it is like an adjustment of the strike price alone. In either case, Chung and Chung's bounds work because they satisfy Corollary 1 of this paper. Thus, Chung and Chang's bounds can be considered special cases of the generalized European claim G in Lemma 4 here. In addition, in this paper, the bounding claim G can be American too. Of course, Chung and Chang do not consider possible stochastic adjustments as in Lemma 4 here.

16. Merton (1973), pp. 154–155, first showed that, for an American call warrant, if the rate of increase in the strike price is less than the interest rate, then a premature exercise is not optimal. Accordingly, the American warrant value will equal the European warrant value. However, he did not use this result to establish upper bounds for call or put options. Also, Merton did not consider adjustments in the number of optioned asset units for this purpose.

17. The value of the standard American option is calculated using a 100-step Binomial tree.

18. Arbitrage condition violations are reported by Ackert and Tian (2000) for the S&P 500 European option contracts, and by Capelle-Blancard and Chaudhury (2001) for the CAC 40 European option contracts.

19. The CBOE launched the OEX contract on March 11, 1983 and the XEO contract on July 23, 2001. Both contracts are cash-settled with a multiple of 100. Since its inception, more than a billion contracts of OEX have been traded. By the close of trading on March 14, 2002, a total of 59,315 OEX traded, of which 27,908 (31,407) are call (put) option contracts. In comparison, a total of 10,776 XEO contracts traded on that day, of which 6,897 (3,879) are call (put) option contracts.

20. That this line of research is promising is demonstrated by the recent work of Chung and Chang (2005). They have extended the theoretical results of Chen and Yeh (2002) and this paper in deriving upper bounds for American options on multiple assets.

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REFERENCES

Ackert, L. F., & Tian, Y. S. (2000). Evidence on the efficiency of index options markets. *Federal Reserve Bank of Atlanta Economic Review*, (First Quarter), 40–52.

Ait-Sahalia, Y., & Lo, A. W. (1998). Nonparametric estimation of state-price densities implicit in financial asset prices. *Journal of Finance*, *53*(2), 499–547.

Andersen, L., & Broadie, M. (2004). A primal-dual simulation algorithm for pricing multi-dimensional American options. *Management Science*, *50*(9), 1222–1234.

Bakshi, G., Cao, C., & Chen, Z. (1997). Empirical performance of alternative option pricing models. *Journal of Finance*, *52*(5), 2003–2049.

Barone-Adesi, G., & Whaley, R. (1987). Efficient analytic approximation of American option values. *Journal of Finance*, *42*(2), 301–320.

Bates, D. (2000). Post-'87 crash fears in the S&P 500 futures option market. *Journal of Econometrics*, *94*, 145–180.

Beber, A., & Brandt, M. W. (2003). *The effect of macroeconomic news on beliefs and preferences: Evidence from the options market*. NBER Working Paper 9914.

Broadie, M., & Detemple, J. (1996). American option valuation: New bounds, approximations, and a comparison of existing methods. *Review of Financial Studies*, *9*(4), 1211–1250.

Broadie, M., Detemple, J., Ghysels, E., & Torres, O. (2000). American options with stochastic dividends and volatility: A nonparametric investigation. *Journal of Econometrics*, *94*, 53–92.

Broadie, M., & Detemple, J. (2004). Option pricing: Valuation models and applications. *Management Science*, *50*(9), 1145–1177.

Bunch, D. S., & Johnson, H. (2000). The American put option and its critical stock price. *Journal of Finance*, *55*(5), 2333–2356.

Capelle-Blancard, G., & Chaudhury, M. (2001). *Efficiency tests of the French index (CAC 40) options market*. McGill Finance Research Centre Working Paper.

Carr, P., Jarrow, R., & Myneni, R. (1992). Alternative characterizations of American put options. *Mathematical Finance*, *2*(2), 87–106.

Chaudhury, M., & Wei, J. (1994). Upper bounds for American futures options: A note. *Journal of Futures Markets*, *14*(1), 111–116.

Chaudhury, M. (1995). Some easy-to-implement methods of calculating American futures option prices. *Journal of Futures Markets*, *15*(3), 303–344.

Chen, R.-R., & Scott, L. (1992). Pricing interest rate futures options with futures-style margining. *Journal of Futures Markets*, *13*, 15–22.

Chen, R.-R., & Yeh, S.-K. (2002). Analytical upper bounds for American option prices. *Journal of Financial and Quantitative Analysis*, *37*(1), 117–135.

Chung, S.-L., & Chang, H.-C. (2005). *Generalized analytical upper bounds for American option prices*. Working Paper, National Taiwan University.

Flamouris, D., & Giamouridis, D. (2002). Estimating implied PDFs from American options on futures: A new semiparametric approach. *Journal of Futures Markets*, *22*(1), 1–30.

Galeti, G., & Melick, W. (2002). *Central bank intervention and market expectations*. BIS Papers No. 10, Monetary and Economic Department, Bank of International Settlements.

Geske, R., & Johnson, H. (1984). The American put option valued analytically. *Journal of Finance*, *39*, 1511–1524.

Grundy, B. (1991). Option prices and the underlying asset's return distribution. *Journal of Finance*, *46*(3), 1045–1069.

Huang, J. (2004). Option pricing bounds and the elasticity of the pricing kernel. *Review of Derivatives Research*, *7*, 25–51.

Huang, J.-z., Subrahmanyam, M. G., & Yu, G. G. (1996). Pricing and hedging American options: A recursive integration method. *Review of Financial Studies*, *9*, 277–300.

Jacka, S. D. (1991). Optimal stopping and the American put. *Mathematical Finance*, *1*(2), 1–14.

Jamshidian, F. (1992). An analysis of American option. *Review of Futures Markets*, *11*(1), 72–82.

Johnson, H. (1983). An analytic approximation for the American put price. *Journal of Financial and Quantitative Analysis*, *18*(1), 141–148.

Ju, N. (1998). Pricing an American option by approximating its early exercise boundary as a multipiece exponential function. *Review of Financial Studies*, *11*(3), 627–646.

Kim, I. J. (1990). The analytic valuation of American options. *Review of Financial Studies*, *3*(4), 547–572.

Leahy, M. L., & Thomas, C. P. (1996). *The sovereignty option: The Quebec referendum and market views on the Canadian dollar*. International Finance Discussion Paper No. 555, Board of Governors of the Federal Reserve System.

Lieu, D. (1990). Option pricing with futures-style margining. *Journal of Futures Markets*, *10*, 327–338.

Lo, A. (1987). Semiparametric upper bounds for option prices and expected payoffs. *Journal of Financial Economics*, *19*, 373–388.

MacMillan, L. (1986). Analytical approximation for the American put option. *Advances in Futures and Options Research*, *1*, 119–139.

Margrabe, W. (1978). The value of an option to exchange one asset for another. *Journal of Finance*, *33*(1), 177–186.

Melick, W. R., & Thomas, C. P. (1997). Recovering an asset's implied PDF from option prices: An application to crude oil during the Gulf crisis. *Journal of Financial and Quantitative Analysis*, *32*(1), 91–115.

Merton, R. (1973). Theory of rational option pricing. *Bell Journal of Economics and Management Science*, *4*, 141–183.

Omberg, E. (1987). The valuation of American puts with exponential exercise policies. *Advances in Futures and Options Research*, *2*, 117–142.

Perrakis, S., & Ryan, P. J. (1984). Option pricing bounds in discrete time. *Journal of Finance*, *39*, 519–525.

Rogers, L. C. G. (2002). Monte Carlo valuation of American options. *Mathematical Finance*, *12*, 271–286.

Soderlind, P., & Svensson, L. E. O. (1997). New techniques to extract market expectations from financial instruments. *Journal of Monetary Economics*, *40*(2), 373–429.