Mediation Analysis Using State-Space Modelling in Single-Case Experimental Design

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Introduction

Mediation analysis: the consideration of how a third variable affects the relation between two other variables (MacKinnon, 2008).

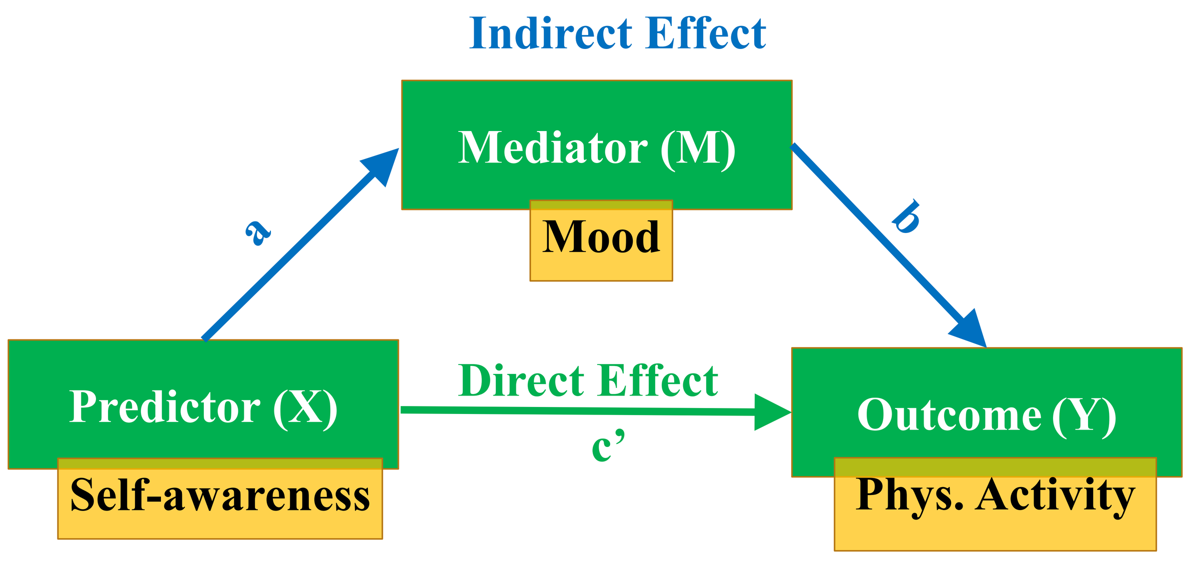

Fig. 1. Path diagram of the mediation analysis model

Example: Self-awareness influences the level of physical activity, and this effect is mediated by mood strength.

State-space modeling (SSM): a modeling framework designed to analyze dynamic systems observed (Shumway and Stoffer, 2000).

Why SSM in Mediation Analysis?

Structural equation modelling (SEM) Framework:

- Focus on cross-sectional between-subject design, with homogeneity assumption.
- Unable to capture longitudinal change when the longitudinal data are intensive (Chow, et al. 2010; Molenaar, 2003).

State-Space Modelling Framework:

- A more accurate and flexible representation in longitudinal within-subject designs with capturing intraindividual variations (Chow, et al. 2010).
- Particularly suitable in Single-Case Experimental Design (SCED).

Model Formulation: State and Measurement Equations

Measurement/Observation Equation:

$$\eta_t = B\eta_{t-1} + \zeta_t, \zeta_t \sim \text{MVN}(0, \Psi)$$

State/Transition Equation:

$$y_t = \Lambda\eta_t + \epsilon_t, \epsilon_t \sim \text{MVN}(0, \Theta)$$

Estimation Method: Kalman Filter

- A recursive algorithm used to estimate the hidden states (latent variables).
- Operate in two primary phases: prediction and update.

Prediction Phase

$$\hat{\eta}_{t|t-1} = B\hat{\eta}_{t-1} + \zeta_t \qquad P_{t|t-1} = BP_{t-1}B' + \Psi$$

Update Phase

$$\epsilon_t = (y_t - \Lambda\hat{\eta}_{t|t-1}) \qquad \hat{\eta}_t = \hat{\eta}_{t|t-1} + K_t\epsilon_t$$

$$K_t = P_{t|t-1}\Lambda'(\Lambda P_{t|t-1}\Lambda' + \Theta)^{-1} \qquad P_t = (I - K_t\Lambda_t)P_{t|t-1}$$

Reference

Chow, S. M., Ho, M. H. R., Hamaker, E. L., & Dolan, C. V. (2010). Equivalence and differences between structural equation modeling and state-space modeling techniques. *Structural Equation Modeling*, *17*(2), 303-332.
MacKinnon D. P. (2008). *Introduction to statistical mediation analysis*. Erlbaum.
Molenaar P. C. (2003). *State space techniques in structural equation modeling*. Retrieved from the Pennsylvania State Department of Human Development and Family Studies
website: http://www.hhdev.psu.edu/hdfs/faculty/docs/StateSpaceTechniques.pdf
Shumway, R. H., & Stoffer, D. S. (2000). *Time series analysis and its applications* (Vol. 3). New York: springer.

Example 1: A Basic Lag-One Example

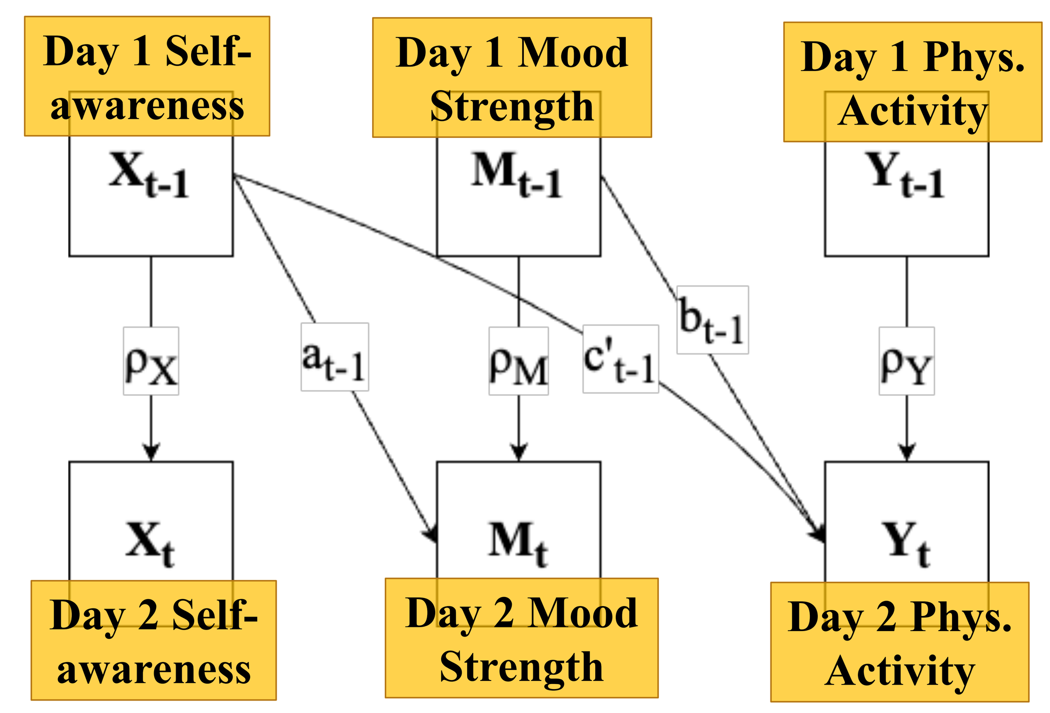

Fig. 2. Path Diagram of the Basic Lag-One SSM Model

$$\begin{pmatrix} X_t \\ M_t \\ Y_t \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} X_t \\ M_t \\ Y_t \end{pmatrix}, \text{where } \Theta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} X_t \\ M_t \\ Y_t \end{pmatrix} = \begin{pmatrix} \rho_X & 0 & 0 \\ a_{t-1} & \rho_M & 0 \\ c'_{t-1} & b_{t-1} & \rho_Y \end{pmatrix}\begin{pmatrix} X_{t-1} \\ M_{t-1} \\ Y_{t-1} \end{pmatrix} + \begin{pmatrix} \zeta_X \\ \zeta_M \\ \zeta_Y \end{pmatrix}, \text{where } \Psi = \begin{pmatrix} \psi_X & 0 & 0 \\ 0 & \psi_M & 0 \\ 0 & 0 & \psi_Y \end{pmatrix}$$

Simulation Studies on Example 1

	True Values	T = 50	T = 100	T = 150
ρ_X	0.6	0.538 (0.118)	0.461 (0.089)	0.580 (0.067)
ρ_M	0.4	0.390 (0.115)	0.344 (0.080)	0.343 (0.069)
ρ_Y	0.5	0.606 (0.065)	0.465 (0.061)	0.559 (0.041)
a_{t-1}	0.4	0.398 (0.098)	0.452 (0.080)	0.328 (0.060)
b_{t-1}	0.4	0.320 (0.111)	0.391 (0.086)	0.385 (0.062)
c'_{t-1}	0.5	0.566 (0.089)	0.510 (0.081)	0.501 (0.052)
ψ_X	0.5	0.656 (0.133)	0.501 (0.071)	0.478 (0.055)
ψ_M	0.4	0.342 (0.069)	0.357 (0.051)	0.359 (0.042)
ψ_Y	0.3	0.253 (0.051)	0.361 (0.051)	0.251 (0.029)

	True Values	T =200	T = 500	T = 1000
ρ_X	0.6	0.542 (0.060)	0.603 (0.036)	0.646 (0.024)
ρ_M	0.4	0.360 (0.059)	0.395 (0.035)	0.372 (0.024)
ρ_Y	0.5	0.523 (0.042)	0.483 (0.025)	0.504 (0.017)
a_{t-1}	0.4	0.347 (0.052)	0.434 (0.034)	0.436 (0.023)
b_{t-1}	0.4	0.345 (0.058)	0.433 (0.035)	0.392 (0.024)
c'_{t-1}	0.5	0.465 (0.051)	0.488 (0.032)	0.476 (0.021)
ψ_X	0.5	0.513 (0.051)	0.492 (0.031)	0.501 (0.022)
ψ_M	0.4	0.367 (0.037)	0.403 (0.026)	0.372 (0.017)
ψ_Y	0.3	0.320 (0.032)	0.303 (0.019)	0.287 (0.013)

Table 1. The Estimates with Standard Errors for Simulated Data Using Lag-One Model

Example 2: A Lag-Two Model

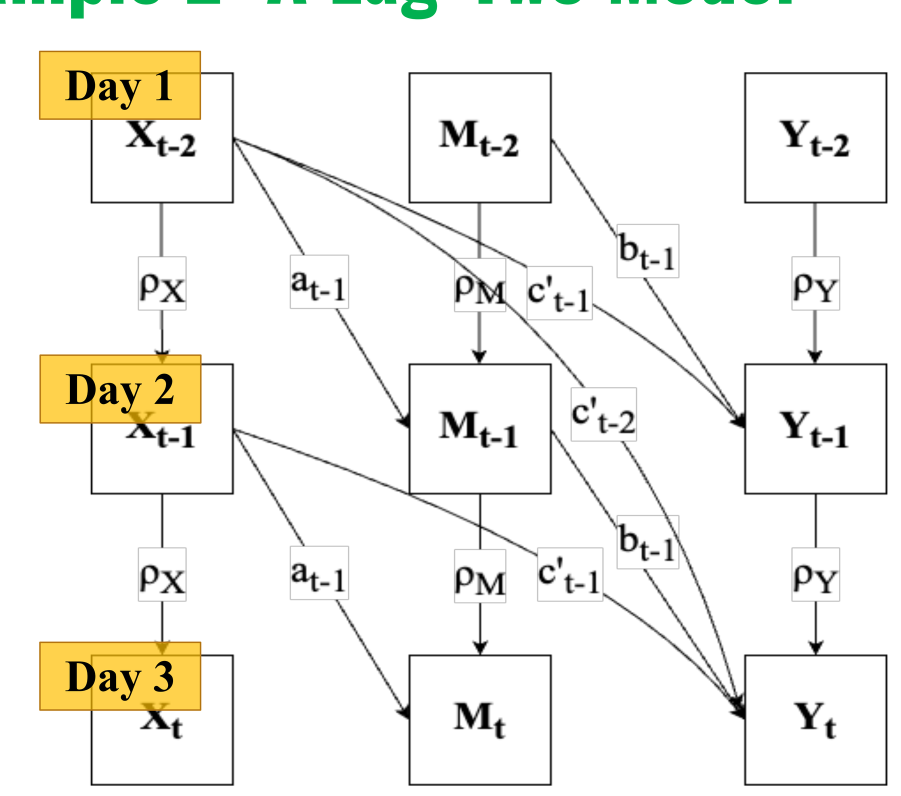

Fig. 3. Path Diagram of the Lag-Two SSM Model

$$\begin{pmatrix} X_t \\ M_t \\ Y_t \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} X_t \\ M_t \\ Y_t \\ X_{t-1} \\ M_{t-1} \\ Y_{t-1} \end{pmatrix}, \text{where } \Theta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} X_t \\ M_t \\ Y_t \\ X_{t-1} \\ M_{t-1} \\ Y_{t-1} \end{pmatrix} = \begin{pmatrix} \rho_X & 0 & 0 & 0 & 0 & 0 \\ a_{t-1} & \rho_M & 0 & 0 & 0 & 0 \\ c'_{t-1} & b_{t-1} & \rho_Y & c'_{t-2} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} X_{t-1} \\ M_{t-1} \\ Y_{t-1} \\ X_{t-2} \\ M_{t-2} \\ Y_{t-2} \end{pmatrix} + \begin{pmatrix} \zeta_X \\ \zeta_M \\ \zeta_Y \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\text{,where } \Psi = \begin{pmatrix} \psi_X & 0 & 0 & 0 & 0 & 0 \\ 0 & \psi_M & 0 & 0 & 0 & 0 \\ 0 & 0 & \psi_Y & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Example 3: A Lag-one SSM with Latent Variables

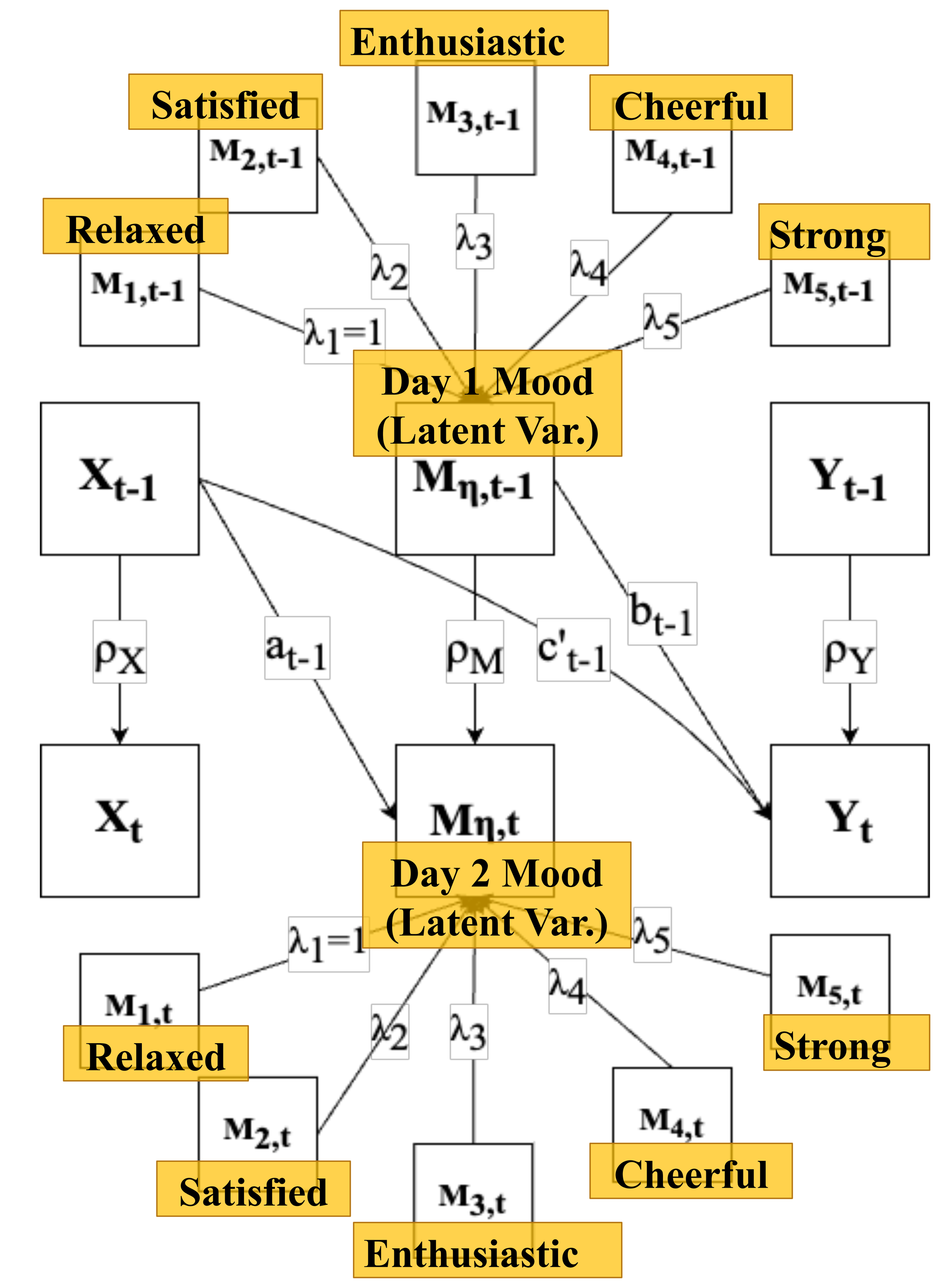

Fig. 4. Path Diagram of a SSM Model with a Latent Mediator Measured by Five Manifest Variables

$$\begin{pmatrix} X_t \\ M_{\eta,t} \\ Y_t \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & \lambda_3 & 0 \\ 0 & \lambda_4 & 0 \\ 0 & \lambda_5 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} X_t \\ M_{1,t} \\ M_{2,t} \\ M_{3,t} \\ M_{4,t} \\ M_{5,t} \\ Y_t \end{pmatrix} + \begin{pmatrix} 0 \\ \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} X_t \\ M_{\eta,t} \\ Y_t \end{pmatrix} = \begin{pmatrix} \rho_X & 0 & 0 \\ a_{t-1} & \rho_{M\eta} & 0 \\ c'_{t-1} & b_{t-1} & \rho_Y \end{pmatrix}\begin{pmatrix} X_{t-1} \\ M_{\eta,t-1} \\ Y_{t-1} \end{pmatrix} + \begin{pmatrix} \zeta_X \\ \zeta_{M\eta} \\ \zeta_Y \end{pmatrix},$$

$$\text{where } \Theta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \theta_1 & 0 \\ 0 & \theta_2 & 0 \\ 0 & \theta_3 & 0 \\ 0 & \theta_4 & 0 \\ 0 & \theta_5 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \Psi = \begin{pmatrix} \psi_X & 0 & 0 \\ 0 & \psi_{M\eta} & 0 \\ 0 & 0 & \psi_Y \end{pmatrix}$$

Discussion

Lag-P Model: Broadly, we can extend the lag-2 SSM model in Example 2 to lag-p, where η_t is influenced by previous lags up to η_{t-1}

$$\eta'_t = B_1\eta'_{t-1} + B_2\eta'_{t-2} + \cdots + B_p\eta'_{t-p} + \zeta_t, \text{ where } \zeta_t \sim \text{MVN}(0, \Psi)$$

$$\eta_t = \begin{pmatrix} \eta'_t \\ \eta'_{t-1} \\ \vdots \\ \eta'_{t-p} \end{pmatrix}, B = \begin{pmatrix} B_1 & B_2 & \dots & B_p \\ I_3 & 0 & \dots & 0 \\ 0 & I_3 & \dots & 0 \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}, \zeta = \begin{pmatrix} \zeta'_t \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \Psi = \begin{pmatrix} \psi'_t & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

Covariates: SSM's flexibility permits the inclusion of covariates (input/control variables) in both measurement and state equations, as described below.

$$\eta_t = B\eta_{t-1} + Cf_t + \zeta_t, \zeta_t \sim \text{MVN}(0, \Psi)$$

$$y_t = \Lambda\eta_t + Ax_t + \epsilon_t, \epsilon_t \sim \text{MVN}(0, \Theta)$$

Examples: Social Interactions, Health Status Changes

Missing Values: With the Expectation-Maximization (EM) algorithm, SSM excels at handling missing data by iterating between estimating the missing data given the model parameters (E-step) and optimizing the model parameters given the estimated data (M-step).

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