Phil 411 *Topics in the Philosophy of Mathematics* Autumn 2018

Mondays, Wednesdays: 13:05 to 14:25 in 688 Sherbrooke, Room 491.

Course Page: MyCourses

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Summary.

This course is one of several designed to act indirectly as an introduction to some of the important issues in the philosophy and history of mathematics. There are many basic philosophical questions about mathematics. There are, first, matters of metaphysics: What is mathematics about? Does it have a subject matter, and if so, what is it? For instance, what are numbers, sets, points, lines, functions, and so on? Or is mathematics merely formal and about nothing in particular, like logic? There are also related semantic matters: What do mathematical statements mean, and can they be true? If so, what is the nature of mathematical truth? And there are, too, matters of epistemology: How is mathematics known? What is its methodology? Is observation involved, or is it a purely mental exercise? In particular, what is proof, and what is *a* proof? Are proofs absolutely certain, immune from rational doubt? Are they thought-experiments, tests, which might suggest or which might be wrong or misleading? Or are they just 'gas', as one famous mathematician called them? If they are a root to knowledge, do proofs constitute the only way we can know mathematical truths? And a question which becomes important in the light of Gödel's Incompleteness Theorems: are there unknowable (unprovable?) mathematical truths?

To approach (some of) these matters historically means to approach (some of) them (often indirectly) by looking at some central developments which shaped modern mathematics, for instance the growth of what we now know as real analysis (through Cartesian analytic geometry, then the differential and integral calculus, and then the modern characterisation of the limit notions and the real numbers), the introduction of the complex numbers (following on the use of the so-called 'imaginary numbers', and the consequent proof of the Fundamental Theorem of Algebra and then complex analysis), the gradual realisation of the importance and inevitability of the theoretical treatment of infinity, and the development in the 19th century of what became known as the axiomatic method. All of this had a profound effect on what we take mathematics to be about, as did the gradual discovery of more and more interconnections between all of these strands. To all of this we have to add the recognition of the importance of 'rigour' in mathematics and with it the recognition of the importance of formal logic on all of this, plus the discovery of the paradoxes and the effect this had on virtually all of these developments. Only with an understanding of these developments does the importance of the epochal work of Gödel in the 1930s fully stand out.

It's an interesting (side) question whether there can be revolutions in mathematics, akin to the Copernican Revolution in astronomy, or the Newtonian Revolution in physics. Perhaps the best candidate for a 'revolution' is the discovery of what became known as *non-Euclidean geometry*, for this had an enormous impact on how we understand what mathematics is about, and on the development of the axiomatic method mentioned above. In his remarkable work *The Elements* (c. 300 BCE), Euclid (from Alexandria in present-day Egypt) set out in 13 Books an axiomatic study of geometry which dominated the study of geometry (and to a lesser extent space) until late into the 19th-c. As one of his five *postulates*, from which (together with certain so-called Common Notions) all the propositions (certainly of plane geometry) are meant to follow, Euclid included his famous Parallel Postulate (EPP), which plays a crucial role in the development of his system, above all in showing that the angle-sum in any triangle is two right-angles (180° — we'll call this the Angle-Sum Theorem or AST) and that there can actually be rectangles and squares. From early on, apparently, the postulate was not taken as having the same degree of evidence as the others, and attempts were made, especially in the modern era, to prove it from the other postulates (or from these together with a with a more plausible substitute for EPP), thus showing its dispensability while preserving its results. Some of these proofs we will look at, particularly the attempts by Wallis, Saccheri and Legendre. One of the things which which became obvious later was that many of the famous proofs innocently *assume* some principle or other which turns out to be *equivalent* to the EPP (and there are many such). One central method (pursued most prominently by the French mathematician, Legendre) was to try to prove AST independently of assuming EPP; it was then taken that this is enough to give us PP, AST being taken as equivalent to Euclid's PP (it nearly is, but not quite!). Speaking quite generally, in principle the angle-sum in a triangle can be either $< 180°$ or $= 180°$ (which is what the AST asserts) or $> 180°$. If we can show that the assumptions that it is $< 180°$ or $> 180°$ lead respectively to contradictions, then the remaining possibility (i.e., the AST) will be proved by *reductio*, thus (it was assumed) yielding EPP. It was fairly easily proved that the assumption that the angle-sum is $> 180°$ will lead to a contradiction, but no contradiction could be derived from the assumption that $< 180°$. This led *de facto* to a geometry in which $\triangle$s have angle-sum $< 180°$, and not $= 180°$, so a non-Euclidean geometry, a geometry in which there are generally no similar (non-congruent) triangles, where the area of a triangle is related to the degree to which its angle-sum falls short of 180°, there are no rectangles, and so on. This geometry was developed above all by Gauss (c. 1800, but unpublished), and then Bolyai (1832) and Lobachevsky (1820s on). (Time permitting, we will look at some of Lobachevsky's presentation.) Nevertheless, it was recognised that the fact that no contradiction had *yet* appeared does not mean no contradiction *can* appear. This then led to the search for 'models' of non-Euclidean geometry, and (in a further step) these were taken to show the consistency of non-E. geometry, i.e., that *no* contradiction is possible. We will look at these models (in particular the so-called 'Poincaré model') and at the structure of the consistency proofs, and what exactly is proved. We will also look at the philosophical conclusions that were drawn from the existence of these models, as well as from the discovery of the geometry itself. We will also look at some of the philosophical literature provoked by the discovery of non-Euclidean geometry, above all in Helmholtz's writings (c. 1865) and in Russell's monograph from 1897 and his exchanges with the mathematician Poincaré (and the latter's own views) on the status of geometry. We will also look at the conclusions drawn from all these developments by David Hilbert as it is represented in the Frege-Hilbert correspondence of 1899 and 1900 and in Hilbert's famous short paper on the number-concept (1900), where important elements in the birth of modern abstract mathematics (and the 'axiomatic method') can be clearly seen.

We will look with some care at the set-up of Euclid's *Elements* (through Heath's translation and extensive commentary), Proclus's discussion and criticism of Euclid's parallel postulate (c. 450 AD/CE), then some of the historical developments subsequently, particularly those of Wallis, Saccheri and Legendre, and finally the philosophical reflections. Some of these developments will be presented directly through the original work, and some through selections from the excellent presentations in books by Ian Mueller, Jeremy Gray, Robin Hartshorne and Marvin Greenberg. All the reading material will be presented as PDFs through the course web site on *MyCourses*.

Prerequisites. Having done PHIL 210 or equivalent is essential, and it would be good if students have done (or plan to do) PHIL 310 (Intermediate Logic) or equivalent and PHIL 311. Having pursued courses in the history of mathematics (e.g., that sometimes offered in the McGill Mathematics Department) would also be an advantage.

Readings. The lectures will concentrate on close reading and discussion of the original texts and readings made available through the *MyCourses* Website. These readings will be *essential*. Many of the lectures will consider these texts in detail, and will assume that they have been read beforehand.

Requirements & grading. Students will be required to attend and participate in class, do the assigned readings, and be prepared to discuss them. The final grade depends on a final class paper (60%) (up to 5000 words), on participation in class (10%), and also on submission of two 'quizzes' for the course, each worth 15%, and spread evenly throughout the course. Each quiz will consist of a series of questions (almost always closely associated with the readings) demanding short answers, and will then ask for a sketch of an essay on one of several questions set. The final essay will generally be an expansion or elaboration of one of your sketch essays.

$\underline{\mathcal{NB}}$: I require that all material (quizzes and final paper) be submitted to me as electronic files in PDF form. Submission will be arranged through the *MyCourses* site.

McGill Policies

1. *McGill University values academic integrity. Therefore all students must understand the meaning and consequences of cheating, plagiarism and other academic offences under the Code of Student Conduct and Disciplinary Procedures (see* `www.mcgill.ca/integrity` *for more information).*

2. *In the event of extraordinary circumstances beyond the University's control, the content and/or evaluation scheme in this course is subject to change.*

3. *Students have the right to submit work in French..*